ON THE FUNCTIONAL POLYSTABILITY OF CERTAIN ESSENTIALLY NONLINEAR NONAUTONOMOUS DIFFERENTIAL SYSTEMS

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For essentially nonlinear differential systems with the limit matrix of coefficients of the first-approximation system, we establish sufficient conditions for functional polystability, which generalizes the notion of exponential polystability.

Introduction

The investigation [1–3] of critical cases of stability of differential systems with slowly varying coefficients enables one to introduce the notion of functional polystability, which generalizes the notion of exponential polystability [4].

The results on functional $Y_{n*}$-stability and polystability are obtained by the methods of generalized “cutting” [1] and nonlinear “frozen” transformations [5], Kostin method [6], and method of Lyapunov functions [7].

1. Statement of the Problem

We investigate a differential system of perturbed motion

$$Y' = F(t, Y),$$  \hspace{1cm} (1)

where $Y = \text{col}(y_1, \ldots, y_n)$, $t \in \Delta \equiv [t_0, \omega], \ t_0 \in \mathbb{R}, \ R \equiv \mathbb{R} \times [-\infty, \omega \leq + \infty], \ F : \Delta \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, $S(Y, r) \equiv \{Y, Y^T : \|Y\| \leq r; \ r \in \mathbb{R}_+\}, \ R_+ \equiv \mathbb{R}_+, \ + \mathbb{R}_+$. $\mathbb{R}^n$ is the $n$-dimensional real Euclidean space,

$$F = \pi PY + G, \ \pi : \Delta \Rightarrow R_+, \ P = \|p_{sk}\|, \ p_{sk} \in C_\Delta^{(h)} \ p_{sk} = p_{sk}^0 + \omega_{sk}(1),$$

$$p_{sk}^0 = \omega_{sk}(1), \ t \uparrow \omega, \ p_{sk}^0 \in \mathbb{R}, \ s, k = 1, n, \ l \in \{1, h\}, \ h \in \mathbb{N},$$

and $\mathbb{N}$ is the set of natural numbers.

The equation $\det(p_0 - \lambda E_n) = 0$, $p_0 = \|p_{sk}\|$, $s, k = 1, n$, has $n_0$ roots $\lambda_0$ satisfying the condition $Re\lambda_0 = 0$, and the other roots $\lambda^*$ of this equation satisfy the condition $Re\lambda^* \neq 0$. Further,

$$G = \sum_{\|Q\| = 2} m G_Q Y^Q + R_m,$$

$$G_Q = \text{col}(G_1 Q, \ldots, G_n Q), \ G_k Q \in C_\Delta^{(h)} \ Q = (q_1, \ldots, q_n), \ q_k \in \{0\} \cup \mathbb{N},$$

$$Y^Q = \prod_{k=1}^n y_k^{q_k}, \ \|R_m\| \leq L \|X\|^{m + \alpha}, \ L : \Delta \Rightarrow \{0\} \cup \mathbb{R}_+, \ L \in C_\Delta, \ m \in \mathbb{N} \setminus \{1\}, \ \alpha \in \mathbb{R}_+. $$
Assume that the equilibrium state \( Y = \overline{0}, \overline{0} \equiv (0, \ldots, 0) \), is unique for the differential system (1) for \( Y \in S(Y, r) \).

Below, we use the following notation and definitions: \( X = \text{col}(x_1, \ldots, x_n), \) \( X \equiv \text{col}(X_{n0}, X_{n-n0}) \equiv \text{col}(X_n, \ldots, X_{n_k}), \) \( X_k = \text{col}(x_{k1}, \ldots, x_{kn}), \) \( X_k \) is a subvector of the vector \( X, \)

\[
\|X_k\| = \left( \sum_{s=1}^{k} |x_{sk}|^2 \right)^{1/2}, \quad \|X\| = \left( \sum_{s=1}^{n} |x_{s}|^2 \right)^{1/2} = \left( \sum_{k=1}^{k_0} \|X_{nk}\|^2 \right)^{1/2},
\]

\( E_k \) and \( H_k \) are, respectively, the identity matrix and translation matrix of dimension \( k \times k, \) \( P_{s,k} \) is an \( (s \times k) \) matrix, \( P_{s,s} = P_s. \)

\[
\|A\| = \left( \sum_{s,k=1}^{n} |a_{sk}|^2 \right)^{1/2}, \quad A = \|a_{ss}\|, \quad s, k = 1, \ldots, n,
\]

\[
X^{-1} = \text{col}(x_1^{-1}, \ldots, x_n^{-1}), \quad \langle X, Y \rangle = \sum_{k=1}^{n} x_k y_k, \quad XY = \text{col}(x_1 y_1, \ldots, x_n y_n),
\]

\[
\Lambda = \max \{f_k: \Delta \Rightarrow R, \ k = 1, \ldots, n \},
\]

if \( \Lambda: \Delta \Rightarrow R_+, \) then \( \Lambda^{-1} f_k = c_k + o_k(1), \) \( t \uparrow \omega, \ c_k \in R, \ k = 1, \ldots, n, \)

\[
\|V(t, x)\| = \text{col} \left( \frac{\partial V}{\partial x_1}, \ldots, \frac{\partial V}{\partial x_n} \right),
\]

\[
R_+ = ]-\infty, 0[, \quad S^*(X; \rho_1, \rho_2) = \{X: \rho_1 \leq \|X\| \leq \rho_2: \rho_1, \rho_2 \in R_+\},
\]

and \( C \) is the set of complex numbers.

**Definition 1.** The equilibrium state \( Y = \overline{0} \) of the differential system (1) is called functionally \( Y_{n_s} \)-stable (in the small) if there exist \( f_n: \Delta \Rightarrow R_+, \ f_n(t) = o(1), \ t \uparrow \omega, \) and, for any arbitrarily small \( \varepsilon \in R_+, \) there exist \( \delta_\varepsilon \in ]0, \varepsilon[, \ T_\varepsilon \in \Delta, \) and \( Y_{n_s} = Y_{n_s}(t; T_\varepsilon, Y_0) \) such that the subvectors of an arbitrary solution \( Y(t; T_\varepsilon, Y_0), Y_0 \equiv Y(T_\varepsilon; Y_0), \) of the differential system (1) satisfy the following inequality for all \( t \in [T_\varepsilon, \omega[ : \)

\[
\|Y_{n_s}(t; T_\varepsilon, Y_0)\| \leq \varepsilon f_{n_s}(t),
\]

where \( \|Y_0\| \leq \delta_\varepsilon \) and \( \|Y\|^2 - \|Y_{n_s}\|^2 < +\infty, \) \( 1 \leq s \leq k_0. \)

**Definition 2.** The equilibrium state \( Y = \overline{0} \) of the differential system (1) is called functionally polystable (in the small) if there exist \( r_s \in R_+, \ s = 1, \ldots, k_0, \) and \( f: \Delta \Rightarrow R_+, \ f = o(1), \ t \uparrow \omega, \) and, for any arbitrarily small \( \varepsilon \in R_+, \) there exist \( \delta_\varepsilon \in ]0, \varepsilon[, \ T_\varepsilon \in \Delta, \) and \( Y_{n_s} = Y_{n_s}(t; T_\varepsilon, Y_0) \) such that the subvectors of an arbitrary solution \( Y(t; T_\varepsilon, Y_0), Y_0 \equiv Y(T_\varepsilon; Y_0), \) satisfy the inequality