MULTIPLICITY RESULTS FOR A FOURTH-ORDER BOUNDARY VALUE PROBLEM

Ma Ruyun (马如云) Ma Qinsheng (马勤生)

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Abstract

This paper deals with multiplicity results for nonlinear elastic equations of the type

\[ u^{(4)} - a_1 u + \beta_1 u^3 + g(x, u, u') = \varepsilon, \quad 0 < x < 1 \]

\[ u(0) = u'(0) = u''(1) = u'''(1) = 0 \]

where \( \varepsilon \in L^1(0, 1) \), \( g: [0, 1] \times R \times R \to R \) is a bounded continuous function, and the pair \( \varepsilon, f \) satisfies

\[ a_1 + (0 + 0.5) \pi^2 \beta_1 = (0 + 0.5) \pi^4 \]

and

\[ a_1 + (k + 0.5) \pi^2 \beta_1 \neq (k + 0.5) \pi^4, \quad \text{for all} \quad k \in N \]

Key words: elastic beam, two-parameter eigenvalue problem, multiplicity result

1. Introduction

The static deformations of an elastic beam with one of its end simply supported and the other end clamped by sliding clamps are described by the following fourth-order boundary value problem

\[ y^{(4)} + f(x) y = \varepsilon(x), \quad 0 < x < 1 \]  
\[ y(0) = y''(0) = y'(1) = y'''(1) = 0 \]  

In [3, 4], Gupta studied the following nonlinear analogue of the boundary value problem (1.1)-(1.2)

\[ y^{(4)} - f(x, y, y', y'', y''') = \varepsilon(x), \quad 0 < x < 1 \]  
\[ y(0) = y''(0) = y'(1) = y'''(1) = 0 \]  

He proved several existing theorems under conditions on \( f \) that are related to the linear eigenvalue problem

\[ y^{(4)} - a y = 0 \]  
\[ y(0) = y''(0) = y'(1) = y'''(1) = 0 \]  

Department of Mathematics, Northwest Normal University, Lanzhou 730070, P. R. China
and
\[
\begin{align*}
  y'''(x) + \beta y''(x) &= 0 \\
  y(0) &= y''(0) = y'(1) = y'''(1) = 0
\end{align*}
\]  

(1.7)

(1.8)

In this paper we study the following nonlinear problem
\[
\begin{align*}
  y'''(x) - \alpha_1 y + \beta_1 y'' + g(x, y, y') &= e, \quad 0 < x < 1 \\
  y(0) &= y''(0) = y'(1) = y'''(1) = 0
\end{align*}
\]  

(1.9)

(1.10)

where \( e \in L^2(0,1) \), \( g : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) is a bounded continuous function, and the pair \((\alpha, \beta)\) satisfies
\[
\alpha_1 + (0 + 0.5)^2 \pi^2 \beta_1 = (0 + 0.5)^4 \pi^4
\]

(1.11)

and
\[
\alpha_1 + (k + 0.5)^2 \pi^2 \beta_1 \neq (k + 0.5)^4 \pi^4, \quad \text{for all} \ k \in \mathbb{N}
\]

(1.12)

We obtain results on nonexistence, existence and multiplicity of solution of (1.9)-(1.10). Our method combines the well-known Lyapunov-Schmidt procedure with a connectivity properties of the solution set of parametrized families of compact vector fields.

In Section II of this paper we study the two-parameter eigenvalue problem
\[
\begin{align*}
  y'''(x) - \alpha y + \beta y'' &= e, \quad 0 < x < 1 \\
  y(0) &= y''(0) = y'(1) = y'''(1) = 0
\end{align*}
\]  

(1.13)

(1.14)

which generalizes (1.5)-(1.6) and (1.7)-(1.8). In Section III we state the main result, while in Section IV we provide the proof.

II. Two-Parameter Eigenvalue Problem

We begin this section by solving the eigenvalue problem (1.13)-(1.14). A pair \((\alpha, \beta)\) such that (1.13)-(1.14) possesses a nontrivial solution will be called an eigenvalue pair. A corresponding nontrivial solution will be eigenfunction.

**Proposition 2.1** \((\alpha, \beta)\) is an eigenvalue pair of (1.13)-(1.14) if and only if
\[
\alpha + (k + 0.5)^2 \pi^2 \beta = (k + 0.5)^4 \pi^4
\]

(2.1)

for some \( k \in \mathbb{N} \cup \{0\} \).

**Proof** Define a linear operator \( F : D(F) \to L^2(0,1) \) by setting
\[
D(F) = \{ u \in L^2(0,1) \mid u, u' \in AC(0,1), \ u'' \in L^2(0,1), \ u(0) = u'(1) = 0 \}
\]

(2.2)

and for \( u \in D(F) \)
\[
F(u) = u''
\]

(2.3)

(Here \( AC[0,1] \) denotes the space of real value absolutely continuous functions on \([0,1]\).) Then
\[
y''' + \beta y'' - \alpha y = (F + r_1) (F + r_2) y
\]

For some \( r_1, r_2 \in \mathbb{C} \), it is easy to see that if (1.13)-(1.14) possesses a nontrivial solution, then either \( r_1 = (k + 0.5)\pi \) or \( r_2 = (k + 0.5)\pi \) for some \( k \in \mathbb{N} \cup \{0\} \). In either case, \( \sin(k + 0.5)\pi x \) is a nontrivial solution of (1.13)-(1.14). By substituting this solution into (1.13), (2.1) follows. Conversely, if (2.1) holds, then clearly \( \sin(k + 0.5)\pi x \) is a nontrivial solution of (1.13)-(1.14).