PRESENTATIONS FOR CRYSTALLOGRAPHIC
COMPLEX REFLECTION GROUPS

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Abstract. We find presentations for the irreducible crystallographic complex reflection groups $W$ whose linear part is not the complexification of a real reflection group. The presentations are given in the form of graphs resembling Dynkin diagrams and very similar to the presentations for finite complex reflection groups given in [2]. As in the case of affine Weyl groups, they can be obtained by adding a further node to the diagram for the linear part. We then classify the reflections in the groups $W$ and the minimal number of them needed to generate $W$, using the diagrams. Finally we show for more than half of the infinite series that a presentation for the fundamental group of the space of regular orbits of $W$ can be derived from our presentations.

1. Introduction

There has been some interest in finding “nice” presentations for the finite irreducible (complex) reflection groups, starting with the work of Coxeter for real reflection groups, then continued by Coxeter and Shephard for certain finite complex reflection groups. A complete list of presentations for the irreducible finite complex reflection groups was given in [2]. Here we give presentations for irreducible infinite discrete complex reflection groups. These groups were classified by Popov [8]. In the real case, one finds the affine Weyl groups. Presentations for these can be obtained from the so-called extended Dynkin diagrams.

It turns out that an analogous construction works in the complex case: Hughes [6] defined extended root-diagrams for certain classes of irreducible finite complex reflection groups. For these groups, our presentations can be

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derived from the latter diagrams by adding a single further relation. In the general case, the infinite irreducible complex reflection group $W$ is presented with a Coxeter-type diagram that contains at most two more nodes than the diagram in [2] presenting the finite linear part $\text{Lin}(W)$ of $W$.

Given the presentations, it is easy to derive some further properties of the infinite irreducible complex reflection group in question, like the number of classes of reflections and the structure of $W/W'$.

In the final section we show that at least in some cases our presentation for an irreducible infinite discrete complex reflection group $W$ allows to obtain a presentation for the fundamental group of the space of regular orbits of $W$. A corresponding result is well known in the finite real case by work of Brieskorn and Deligne, in most finite complex cases by [3], and in the real affine case by Viêt Dũng. It would be interesting to prove this in general, and without case-by-case arguments. This would in particular require an a-priori way to define the nice presentation of a complex reflection group.

2. Crystallographic complex reflection groups

The finite complex reflection groups have been classified by Shephard and Todd. Nice presentations for these were obtained in [1] and [2], completing earlier work of Coxeter, Shephard and others. Here we are interested in the infinite affine reflection groups. The following setup is taken from the lecture notes [8] of Popov.

Let $E$ be the $n$-dimensional affine space over $\mathbb{C}$, and $V$ its space of translations. By choosing a base point $v_0 \in E$ we may identify the group $\text{A}(E)$ of affine transformations of $E$ with the semi-direct product $V \cdot \text{GL}(V)$. For $W$ a subgroup of $\text{A}(E)$, the image of $W$ in the factor group $\text{GL}(V)$ is denoted by $\text{Lin}(W)$; it is independent of the choice of $v_0$. The subgroup of translations of $W$ is denoted by $\text{Tran}(W)$.

We fix a positive definite sesquilinear (i.e., unitary) form $(,)$ on $V$. An element of finite order $a \in \text{A}(E)$ is called an affine reflection if it preserves the distance on $V$ induced by $(,)$ and the codimension of its space of fixed points is equal to 1. A subgroup $W$ of $\text{A}(E)$ is called an r-group if it is discrete and generated by affine reflections.

By [8, Theorem 1.4] every r-group is a direct product of irreducible r-groups, so it suffices to consider the irreducible groups. It is clear that any finite r-group fixes some point of $E$ and thus is equivalent to a finite complex reflection group in the usual sense. The infinite irreducible r-groups $W$ were classified by Popov. There arise two essentially different cases. Either $E/W$ is not compact in which case $W$ is equivalent to the complexification of an affine Weyl group (loc. cit., Theorem 2.2). If $E/W$ is compact, $W$ is called crystallographic. Here quite a number of new cases arise (i.e., cases without real analogue).

The extension of $\text{Tran}(W)$ by $\text{Lin}(W)$ defines an action of $\text{Lin}(W)$ by automorphisms of the lattice $\text{Tran}(W)$, hence a real representation of $\text{Lin}(W)$.