ISOMONODROMIC DEFORMATIONS OF HEUN AND PAINLEVÉ EQUATIONS

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Continuing the study of the relationship between the Heun and the Painlevé classes of equations reported in two previous papers, we formulate and prove the main theorem expressing this relationship. We give a Hamiltonian interpretation of the isomonodromic deformation condition and propose an alternative classification of the Painlevé equations, which includes ten equations.

1. Introduction

Since 1889, when Heun wrote the paper about solutions of a Fuchsian second-order ordinary differential equation (ODE) with four regular singularities [1], this equation has been commonly called the Heun equation. One hundred years later, Seeger, Ronveaux, and Lay organized the Centennial Workshop on the Heun equation [2], which surveyed the modern state of research into solutions of the Heun equation. A main result of the workshop was the recognition that beyond the Heun equation, its confluent cases should be studied. These confluent cases (namely, four of them) were introduced and investigated by Decarreau et al. [3, 4]. A refined classification of equations belonging to the Heun class based on the notion of the $s$-rank of a singularity was proposed later [5, 6]. This classification includes the so-called reduced confluent cases of Heun equations and distinguishes ten types of equations belonging to the Heun class.

In 1898, Painlevé started a series of articles [7] in which he introduced the class of second-order nonlinear ODEs whose solutions have movable singularities that are only poles. This class of equations is now known as the Painlevé equations. The complete list of Painlevé equations was presented by Gambier [8] and by Garnier in a slightly different notation [9]. Particular solutions of Painlevé equations comprise the class of nonlinear special functions of mathematical physics, the so-called Painlevé transcendents.

Even before the list of Painlevé equations was completed, Fuchs [10] found a remarkable relationship between the Heun equation and the Painlevé $P^{VI}$ equation (in the modern classification). He introduced terms related to an additional apparent singularity in the Heun equation. An apparent singularity is a regular singularity in the neighborhood of which the general solution of the corresponding ODE is a holomorphic function. In Fuchs’s studies, the $P^{VI}$ equation arose as a compatibility condition ensuring that the introduced equation and the additional linear equation with the differentiation with respect to a parameter have common solutions. However, the studies of Fuchs were not sufficiently developed. Another link between Painlevé equations and linear equations (namely, linear systems) found by Schlesinger [11] and expressed in modern terms in other publications [12, 13] became popular. The method based on this approach is now known as the isomonodromic deformations method or the method of deformations preserving the monodromy (see, e.g., [14]).

It was proved that the Painlevé equations can be considered as Newtonian equations of motion if the corresponding quantum Hamiltonians stem from Heun equations [15]. This observation explained the previously known Hamiltonian structure of Painlevé equations [14, 16, 17].

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Several problems remained unsolved in [15]. The first was the question of the existence of a one-to-one correspondence between the Heun and Painlevé classes of equations. The second problem (still unsolved in [15]) was an alternative classification of Painlevé equations from the standpoint of the Heun equation theory. These questions were answered in [18]. However, the third problem of establishing the relationship between the isomonodromic deformation approach to Painlevé equations and the Hamiltonian approach [15] to these equations remained. In this paper, we solve this problem using the method in [10] based on deforming the equations of the Heun class by introducing apparent singularity terms in the equation.

We first present the main results in [15]. Each equation of the Heun class can be represented in the form

$$\frac{1}{f(t)} [P_0(z,t)D^2 + P_1(z,t)D + P_2(z,t)] y(z) = \lambda y(z), \quad D := \frac{d}{dz}. \quad (1)$$

In Eq. (1), $P_0(z,t)$, $P_1(z,t)$, and $P_2(z,t)$ are polynomials in $z$, $t$ is the scaling parameter, and $\lambda$ is the accessory parameter. If the quantum observables $\hat{q}$ and $\hat{p}$ (the coordinate and momentum) are the respective multiplication by $z$ and differentiation $D$, then Eq. (1) admits a Hamiltonian structure and can be rewritten as

$$H(\hat{q}, \hat{p}, t)y = \lambda y, \quad (2)$$

where

$$H(\hat{q}, \hat{p}, t) = \frac{1}{f(t)} [P_0(\hat{q}, t)\hat{p}^2 + P_1(\hat{q}, t)\hat{p} + P_2(\hat{q}, t)].$$

In Eq. (2), the function $H$ is the Hamiltonian adiabatically depending on the parameter $t$, which is considered as time, and $\lambda$ is the energy of the system. The corresponding Hamiltonian in classical mechanics is quadratic in the classical momentum $p$,

$$H(q, p, t) = \frac{1}{f(t)} [P_0(q, t)p^2 + P_1(q, t)p + P_2(q, t)]. \quad (3)$$

After the Legendre transformation of this Hamiltonian, the momentum $p$ is replaced by the velocity $q_t$. The corresponding Lagrangian $\mathcal{L}(q, q_t, t)$, which is again quadratic in $q_t$, is

$$\mathcal{L}(q, q_t, t) = \frac{f(t)}{4P_0(q, t)} \left( q_t - \frac{P_1(q, t)}{f(t)} \right)^2 - \frac{P_2(q, t)}{f(t)}. \quad (4)$$

The Euler–Lagrange equation of motion corresponding to this Lagrangian is

$$q_{tt} = \frac{1}{2} \frac{\partial}{\partial q} \left( \log P_0(q, t) \right) q_t^2 - \left( \frac{\partial}{\partial t} \left( \log f(t) \right) - \frac{\partial}{\partial t} \left( \log P_0(q, t) \right) \right) q_t + \frac{P_0(q, t)}{f^2(t)} \left( \frac{\partial^2 P_0^2(q, t)}{\partial q^2} + f(t) \frac{\partial}{\partial t} \frac{P_1(q, t)}{P_0(q, t)} - \frac{2}{\partial P_2(q, t)} \frac{\partial}{\partial q} \right). \quad (5)$$

For any particular equation from the Heun class, Eq. (5) is a Painlevé equation with the Painlevé property (the proof is the direct calculation). This means that movable singularities of all solutions of the equation are poles (no movable branch points or movable essential singularities exist).

In fact, the inverse proposition that any Painlevé equation can be obtained as an equation of classical motion corresponding to an equation belonging to the Heun class was proved in [15]. The calculations in [15, 18] suffice to prove both the direct and inverse propositions.

\footnote{The physical meaning of $t$ can be different in particular problems.}