A. N. Leznov

A new class of integrable mappings and chains is introduced. The corresponding 1+2 integrable systems that are invariant under such integrable mappings are presented in an explicit form. Soliton-type solutions of these systems are constructed in terms of matrix elements of fundamental representations of semisimple $A_n$ algebras for a given group element. The possibility of generalizing this construction to the multidimensional case is discussed.

1. Introduction

In an old paper by the author [1], an effective method based on the two-dimensional zero-curvature condition was proposed for constructing exactly integrable systems in two dimensions and their general solutions. During the last fifteen years, the situation has become more clear and understandable.

It turned out that the method in [1] allows constructing integrable mappings [2, 3] responsible for the existence of various hierarchies of integrable systems. Each equation in a given hierarchy is invariant under the transformation of the corresponding mapping (or substitution).

Moreover, it has become clear that the formalism of the $L-A$ pair is not the principal focus of the whole construction. The integrable mappings can be obtained more directly, and the technique of $L-A$ pairs is only a consequence. The situation arising in the explicit solution of the quantum two-dimensional Toda lattice [4] for Heisenberg operators is the most important argument for the necessity of developing a new approach unrelated to the zero-curvature representation. The method in the present paper works in this case although the $L-A$ representation is completely absent. As a related fact, we recall the general solution of the periodic Toda lattice equations, which can be obtained in the form of an absolutely convergent infinite series using the following construction without any $L-A$ pair formalism [5].

In this paper, we propose the following scheme. At the first step, we introduce two equations of the $S$-matrix type that are integrable by quadratures for two group-valued functions depending on two different arguments. The coefficient functions of these equations are determined by the structure of the corresponding Lie algebra and by the choice of a certain grading in it. We do not differentiate between the algebra and superalgebra cases, recalling only that even (odd) elements of superalgebras are always multiplied by even (odd) elements of the Grassmann space. Using these two group elements, we construct a new composite one. Equivalence relations arising between its matrix elements lead to the integrable substitution.

At the second step, we assume an additional dependence of arbitrary functions determining the general solution of the integrable substitution (with fixed ends) on some "timelike" parameters. This is achieved by two additional equations for the abovementioned group elements $M_A$ such that their self-consistency condition leads to the explicit dependence of the arbitrary functions on both space- and timelike parameters. Using these time parameters, we present the hierarchy of completely integrable systems, each of which is invariant under the transformation of the integrable mapping (constructed at the first step).

At the third and final step (which, in fact, is not closely connected with the previous ones), we observe...
that in the framework of the method [1], there exists a hidden, previously omitted nontrivial possibility for
generalizing the whole construction to the multidimensional case.  

The most remarkable feature of this approach is that the dimension of the possible “multigeneralization” is uniquely determined by the properties of the algebra and the choice of the grading in it. The “multigeneralization” is sometimes equivalent to a trivial change of variables and sometimes leads to new nontrivial possibilities. Nevertheless, in all cases, it is possible to obtain only particular (not general) solutions of systems and equations arising in this way.

This paper is organized as follows. In Sec. 2, we briefly repeat the first section in [1], but in the “opposite direction” compared to the original. In Sec. 3, for convenience of the reader, we present results from the representation theory of semisimple (super) algebras and groups that are most important for further applications. In Sec. 4, we show how to avoid the relatively cumbersome Gauss decomposition procedure (in the cases where it is equivalent to solving the whole problem); we present the integrable systems together with their general solutions in terms of the matrix elements of various fundamental representations of semisimple algebras. We thus construct UToda(m1, m2) integrable mappings or substitutions. In Sec. 5, we demonstrate how to introduce the evolution parameters in order to obtain the hierarchies of integrable systems that are invariant under transformations of the integrable mappings constructed. In Sec. 6, we discuss the possibilities for generalizing this construction to the multidimensional case (with predetermined dimensions!). Concluding remarks are collected in Sec. 7.

2. Moving in the opposite direction

We consider an arbitrary finite-dimensional graded algebra \( \mathcal{G} \). This means that \( \mathcal{G} \) can be represented as a direct sum of subspaces with different grading indices,

\[
\mathcal{G} = \bigoplus_{k=1}^{N_+} \mathcal{G}_{k/2} \oplus \mathcal{G}_0 \oplus \bigoplus_{k=1}^{N_-} \mathcal{G}_{k/2}.
\]  

(1)

Generators with integer grading indices are called bosonic, and those with half-integer grading indices are called fermionic. Positive (negative) grading corresponds to upper (lower) triangular matrices.

Let \( M_+(y) \) and \( M_-(x) \) be elements of the group (when it exists) corresponding to algebra (1) and only some solutions of equations of the S-matrix type in a given finite-dimensional representation of \( \mathcal{G} \):

\[
\frac{\partial M_-}{\partial x} = L_+^{m_1}(x)M_- = \sum_{s=0}^{m_1} A^{-s}(x)M_-, \quad \frac{\partial M_+}{\partial y} = L_+^{m_2}(y)M_+ = \sum_{s=0}^{m_2} B^{+s}(y)M_+,
\]

(2)

where \( A^{-s} \) and \( B^{+s} \) are arbitrary functions of their arguments taking values in the corresponding graded subspaces and \( s \) is an integer or half-integer number.

We introduce the group element

\[
K = M_+ M_-^{-1}
\]

(3)

(all considerations here have meaning only if the group exists). According to [1], \( K \) can be represented in the form of the Gauss decomposition

\[
K = M_+ M_-^{-1} = N_-^{-1} g_0 N_+,
\]

where \( N_\pm \) are elements of the positive (negative) nilpotent subgroups and \( g_0 \) is an element of the group whose algebra is the zero subspace. We consider the group element

\[
G = N_- M_+ = g_0 N_+ M_-
\]

(4)

\[\text{See [6] for possibilities for enlarging the class of integrable systems that were not discussed in [1].}\]