CONTINUUM LIMIT IN THE FERMIONIC HIERARCHICAL MODEL

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We discuss the problem of rigorously constructing the continuum limit in the fermionic hierarchical model. The continuum limit constructed as the limit of fields on the refined hierarchical lattices is a field on a p-adic continuum. We investigate the problem of reconstructing the coupling constants of the continuum model from the coupling constants of the discretized model.

As noted in [1-2], p-adic field-theory models are a natural continuum generalization of the hierarchical Dyson models [3-6]. The p-adic models arise in a wide range of problems in mathematical physics [7, 8].

In this paper, we consider the problem of constructing the continuum limit in the framework of the so-called fermionic hierarchical model (see [9-11]). The corresponding problem for the usual lattice models in quantum field theory is the basic (and complicated) problem of the constructive field theory [12, 13]. It is known that the Gaussian part is fixed in these models (being given, for example, by a lattice analogue of the Laplace operator). The existence of a continuum limit means that there are limits of the correlation functions as the lattice spacing tends to zero; the coupling constants of the intermediate lattice models then depend on the lattice spacing and tend to infinity under the cutoff removal.

A peculiarity of our model is that the Gaussian part is given by the action

\[ H_0(\psi^*; \alpha) = c(\alpha) \int |x - y|^{-\alpha d} (\bar{\psi}_1(x)\psi_1(y) + \bar{\psi}_2(x)\psi_2(y)) \, dx \, dy, \]

where \( x \) and \( y \) are \( d \)-dimensional p-adic arguments, \( dx \) is the corresponding Haar measure, \( \bar{\psi}^*(x) = (\bar{\psi}_1(x), \psi_1(x), \bar{\psi}_2(x), \psi_2(x)) \) is a four-component Grassmann-valued field (whose components are the generating elements of a Grassmann algebra), \( |\cdot| \) is the p-adic norm, \( c(\alpha) \) is a normalization constant, and \( \alpha \) is a real parameter. This Hamiltonian determines the so-called self-similar (scale-invariant) fermionic field with the self-similarity parameter \( \alpha \). The value \( \alpha = 1 + 2/d \) gives rise to a p-adic analogue of the Hamiltonian that is given by the Laplace operator in the real case. We note that in contrast to the real case, the p-adic version with \( \alpha = 1 + 2/d \) describes a model with long-range interaction.

The non-Gaussian part is given by the Hamiltonian

\[ H_1 = r \int (\bar{\psi}_1(x)\psi_1(x) + \bar{\psi}_2(x)\psi_2(x)) \, dx + g \int \bar{\psi}_1(x)\psi_1(x)\bar{\psi}_2(x)\psi_2(x) \, dx. \]

Our model can be viewed as a fermionic analogue of the hierarchical \( \varphi^4 \)-model, its non-Gaussian part simulating the Gross-Neveu model [14, 15]. We also note that a certain degenerate version of the fermionic hierarchical model was considered in [16].

We study the discrete (hierarchical) version of the Hamiltonian \( H_0 + H_1 \). The Gaussian part of the hierarchical model is invariant under transformations of the Kadanoff-Wilson block-spin renormalization group (RG) with the parameter \( \alpha \). The RG action in the coupling-constant space is explicitly evaluated [9], and the inverse mapping also exists. This is the main simplifying point in the problem of constructing the continuum limit in our model.

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In particular, we show that for those values of $\alpha$, $r$, and $g$ considered in [10] that allow the existence of the thermodynamic limit in the hierarchical model, the correlation-function limit as the hierarchical-lattice spacing goes to zero also exists. As follows from the results of [10] and from computer experiments, the existence domain of the thermodynamic limit in the $(r, g)$ plane comprises a full-measure set in $\mathbb{R}^2$ if $\alpha > 1$.

A new feature is that the limit of the lattice field-theory coupling constants as the lattice spacing goes to zero exists for $\alpha > 2$. This limit determines the continuum field theory whose discretization leads to the original hierarchical model.

Let $T$ be the mapping that associates the coupling constants of the limiting continuum theory with the coupling constants $r$ and $g$ of the original hierarchical model. For $\alpha > 2$, the mapping $T$ satisfies the commutation relation

$$TR = ST,$$

where $R$ is the hierarchical RG transformation in the $(r, g)$ plane and $S$ is the diagonal matrix given by the eigenvalues of the differential of $R$ at the origin. It then follows that the mapping $T$ is inverse to the normalizing transformation $P$ defined by the functional equation

$$RP = PS$$

The mapping $P$ can be viewed as a discretization operator that assigns the coupling constants of the discretized field to the coupling constants of the continuum field (see [17] for more details). Therefore, $T = P^{-1}$.

As known from the general theory of normal forms [18], however, the mapping $P$ (and also $P^{-1}$) can be determined not only for $\alpha > 2$ but also for the nonresonance values $\alpha < 2$ (a more precise statement is given below) and is given by a convergent power series in $r$ and $g$ in a neighborhood of the origin. Thus, the definition of $T$ can be extended to the domain $1 < \alpha < 2$ except for a discrete series of values of $\alpha$; $T$ then determines the coupling constants of the continuum theory corresponding to the original hierarchical model. However, these coupling constants cannot be determined via a direct limiting process and, in particular, cannot be determined at resonance values of $\alpha$. A similar picture may also be true for bosonic models in the real case if the kernel of the Gaussian part is given by a power of the Laplace operator, which itself corresponds to the resonance value. We note that a series of resonance values of $\alpha$ for functional equation (2) coincides with the series of ultraviolet poles of the $p$-adic Feynman amplitudes in the continuum version of the model.

We now recall several definitions [9]. Let $Q_p$ be the $p$-adic number field and $| \cdot |_p$ be the $p$-adic norm on $Q_p$. The fractional part of a $p$-adic number $x = c_{-n}p^{-n} + \cdots + c_{-1}p^{-1} + c_0 + c_1p + \ldots$ is denoted by $\{x\}$, i.e.,

$$\{x\} = c_{-n}p^{-n} + \cdots + c_{-1}p^{-1}.$$ 

For $x = (x_1, \ldots, x_d) \in Q_p^d$, we set

$$|x|_p = \max_i |x_i|_p, \quad \{x\} = (\{x_1\}, \ldots, \{x_d\}).$$

Then the discrete set

$$T^d_p = \{x \in Q_p^d : x = \{x\}\}$$

can be viewed as a hierarchical lattice with the elementary cell size $n = p^d$ and with the hierarchical distance $d(i, j) = |i - j|_p$, $i, j \in T^d_p$.

We consider a four-component fermionic field $\psi^*(x) = (\psi_1(x), \psi_1(x), \psi_2(x), \psi_2(x))$, $x \in Q_p^d$. The group of scaling transformations is defined by

$$(S_\lambda(\alpha) \psi^*)(x) = |\lambda|^{(1-\alpha/2)d} \psi^*(\lambda x),$$