FERMI–WALKER TRANSPORT AND THE WEYL CONNECTION

B. M. Barbashov and A. B. Pestov

We derive equations relating the Fermi–Walker and the congruent Weyl transports. Using these equations, we show that a non-Abelian gauge field can result in the Thomas precession of a gyroscope. We find solutions to the equations for such a non-Abelian gauge field.

1. It was shown in [1] that the congruent Weyl transport [2] defines a non-Abelian gauge field. In [3], a spinor representation of the Weyl group was constructed, and the corresponding spinor current that generates the gauge field was found. This raises a question about the macroscopic effects of the gauge fields. One such effect is investigated in this paper. It is shown that if no electromagnetic or gravitational forces act on a rotating body, then its spin precession can be caused by the non-Abelian gauge field.

It is known (e.g., see [4]) that the most general law for the parallel transport of a vector along a curve $x^i = x^i(\tau)$ is defined by the system of ordinary differential equations

$$\frac{dS^i}{d\tau} = -\Gamma^i_{jk} u^j S^k,$$

where $u^i = dx^i/d\tau$. Because the vector length must not change, the coefficients $\Gamma^i_{jk}$ must satisfy the system of algebraic equations

$$\partial_j g_{ik} - \Gamma^l_{ji} g_{lk} - \Gamma^l_{jk} g_{il} = 0.$$

We assume that the metric tensor $g_{ij}$ is given and the inner product is defined by $g(S, S) = g_{ij} S^i S^j$. The general solution to the above system of algebraic equations is

$$\Gamma^i_{jk} = \left\{ \begin{array}{c} i \\ jk \end{array} \right\} + G^i_{jk}, \quad (1)$$

where $G^i_{jk} = g^{ik} G_{jkl}$ and $G_{jkl}$ is a third-rank tensor field antisymmetrical with respect to $k$ and $l$, i.e., $G_{jkl} + G_{kjl} = 0$. This $G_{jkl}$ is the Weyl non-Abelian gauge field whose properties were investigated in [1, 3]. In Eq. (1), $\left\{ \begin{array}{c} i \\ jk \end{array} \right\}$ denotes the Christoffel symbol of the metric tensor $g_{ij}$, i.e.,

$$\left\{ \begin{array}{c} i \\ jk \end{array} \right\} = \frac{1}{2} g^{ik} (\partial_j g_{kl} + \partial_k g_{jl} - \partial_l g_{jk}).$$

Connection (1) defines the congruent Weyl transport, during which the vector field is changed according to the law

$$\frac{dS^i}{d\tau} + \left\{ \begin{array}{c} i \\ jk \end{array} \right\} u^j S^k = -G^i_{jk} u^j S^k. \quad (2)$$

This law includes translation proper to Riemann geometry and rotation determined by the metric tensor $g_{ij}$ and the bivector $G_{jkl} u^j$. The Weyl gauge theory is a differential-geometric realization of the abstract gauge field theory such that the space–time and gauge space are not separated [1]. In this paper, a possible physical manifestation of the Weyl non-Abelian gauge field is discussed.

1Joint Institute for Nuclear Research, Dubna, Moscow Oblast, Russia.

2. We consider a rotating body, e.g., a gyroscope, that accelerates because of a force applied to its center of mass. Such forces have no moment and therefore do not change the vector $S^i$ of the body spin if a rotation in the plane spanned by the velocity four-vector $u^i$ and the acceleration

$$a^i = \frac{\delta u^i}{\delta \tau} = \frac{d^2 x_i}{d\tau^2} + \left\{ \begin{array}{c} i \\ jk \end{array} \right\} \frac{dx^j}{d\tau} \frac{dx^k}{d\tau}$$

are excluded. Such rotation does not change the angle formed by the spin vector $S^i$ and the velocity four-vector $u^i$. From the geometric standpoint, this means that the body spin translation is the Fermi–Walker transport \[5-7\]

$$\frac{dS^i}{d\tau} + \left\{ \begin{array}{c} i \\ jk \end{array} \right\} u^i S^k + (u^i a^j - u^j a^i) S_j = 0. \text{(3)}$$

We assume here that the center-of-mass trajectory is known. Therefore, the velocity $u^i$ and the acceleration $a^i$ are known, and solving Eqs. (3), we can find the angular velocity of the precession.

It is easy to verify that the Fermi–Walker transport does not change the vector lengths and the angles formed by the vectors. Because congruent transport (2) has the same property, we can ask if Eqs. (2) can describe the spin precession in the absence of gravitational and electromagnetic fields. To answer this question, we find the trajectories on which Eqs. (2) and (3) are equivalent. These trajectories satisfy the system of ordinary second-order differential equations

$$u_i a_j - u_j a_i + G_{ijk} u^i = 0 \text{(4)}$$

that follow from considering Eqs. (2) and (3) together. We note that the trajectories in a gravitational field need not satisfy Eqs. (4), because the expression $G_{ijk} u^i$ need not be equal to zero if $a^i = 0$.

We now verify that Eqs. (4) are consistent. Because $u_i u^i = 1$, Eqs. (4) imply the system

$$a^i + G^i_{jk} u^j u^k = 0 \text{(5)}$$

that defines the geodesic lines for connection (1). To verify that Eqs. (2) and (5) imply the conservation of the spin–velocity orthogonality, we derive

$$\frac{\delta S^i}{\delta \tau} u_i = -G_{ijk} u^i S^j u^k$$

from Eqs. (2) and

$$S_i \frac{\delta u^i}{\delta \tau} = G_{ijk} u^i S^j u^k$$

from Eqs. (5). Therefore,

$$\frac{\delta}{\delta \tau}(u_i S^i) = 0,$$

i.e., the spin–velocity orthogonality is conserved. Therefore, Eqs. (2) can be used to describe the spin precession during the motions defined by Eqs. (4). We now investigate Eqs. (4) in detail.

3. It is known that the irreducible quantities with respect to the fundamental group are of principal interest. For the field $G_{ijk}$, such a fundamental group is the Weyl group. It has the same structure as the Lorentz group. The decomposition of $G_{ijk}$ into the irreducible components is \[8\]

$$G_{ijk} = \frac{2}{3} (T_{ijk} - T_{ikj}) + \frac{1}{3} (g_{ij} F_{k} - g_{ik} F_j) + \epsilon_{ijkl} A^l,$$

where \(T_{ijk}\) is the tensor of rank 3.