KDV EQUATION ON A HALF-LINE WITH THE ZERO BOUNDARY CONDITION

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We solve the mixed problem for the KdV equation with the boundary condition \( u|_{z=0} = 0, \quad u_{zz}|_{z=0} = 0 \) using the inverse scattering method. The time evolution of the scattering matrix is efficiently defined from the consistency condition for the spectra of two differential operators giving the L-A pair.

1. Introduction

The inverse scattering method (ISM) makes possible a rather complete investigation of the Cauchy problem for nonlinear integrable equations [1]. In the case of one spatial variable, solving the Cauchy problem reduces to successively solving two nonlinear problems: constructing the fundamental solution of a nonlinear system of ordinary differential equations and then solving the matrix Riemann problem of the analytic conjugation of functions in the complex plane, which is equivalent to a system of singular integral equations. Many applied problems are formulated as mixed problems for nonlinear partial differential equations. The mixed problem for equations like the Korteweg–de Vries (KdV) equation or the nonlinear Schrödinger equation are certainly of practical interest (see, e.g., [2–5]) and has therefore been the subject of much investigation over a period of years. After the series of papers by Its and Fokas [6–8], it became clear that under arbitrary boundary conditions, solving the mixed problem for the above-mentioned equations had not met the same success as solving the Cauchy problem on the whole line. But there is a specific class of boundary conditions that are completely consistent with the integrability property. Under these conditions, the mixed problem is effectively embedded in the ISM scheme. A number of examples of such boundary conditions were discussed in [6, 9–12]. The symmetry test developed in [13, 14] makes possible the total description of the classes of integrable boundary conditions for integrable equations in 1+1 dimensions. For example [15], the KdV equation is consistent with boundary conditions of the type \( u(x,t)|_{z=0} = a, \quad u_{zz}(x,t)|_{z=0} = b, \) where \( a \) and \( b \) are arbitrary constants. The regular and real algebraic-geometric solutions of the KdV equation satisfying such boundary conditions were described in [16]. Integrable boundary conditions are rather restricted as a class but can provide a basis for investigating the influence of a general boundary condition, for example, using perturbation theory methods.

In this paper, we study the mixed problem on a half-line for the KdV equation under arbitrary initial conditions and the simplest boundary conditions \( a = b = 0 \)

\[
\begin{align*}
\frac{u_t}{u_{xx}} &= 6uu_x, \quad x > 0, \quad t > 0, \\
u|_{z=0} &= 0, \quad u_{xx}|_{z=0} = 0, \\
u|_{t=0} &= u_0(x), \quad u_0(x)|_{z \to +\infty} \to 0.
\end{align*}
\]

The initial function \( u_0(x) \) is required to be smooth and rapidly decreasing as well as consistent with boundary condition (2) at the corner point \( u_0(0) = u_{0xx}(0) = 0 \). We show that under these conditions, problem (1)–(3) is effectively embedded in the ISM scheme. We note that problem (1)–(3) is correctly

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formulated. The questions of the existence and the uniqueness of the solution of this problem are well investigated (see, e.g., [17]).

As is known [1], the KdV equation is a consistency condition for the two systems of linear equations in \( x \) and \( t \)

\[
\begin{align*}
Y_x &= U Y, \\
Y_t &= V Y,
\end{align*}
\]

where the coefficient matrices have the forms

\[
U = \begin{pmatrix} 0 & 1 \\ u - \lambda & 0 \end{pmatrix}, \quad V = \begin{pmatrix} u_x & -4\lambda - 2u \\ u_{xx} - (4\lambda + 2u)(u - \lambda) & -u_x \end{pmatrix}.
\]

It is easy to verify that boundary conditions (2) are satisfied if and only if Eq. (5) along the line \( x = 0 \) takes the form

\[
Y_t = \begin{pmatrix} u_x(0, t) & -4\lambda \\ 4\lambda^2 & -u_x(0, t) \end{pmatrix} Y.
\]

In other words, linear system of equations (4)-(6) gives the Lax representation for mixed problem (1)-(3).

The main difference between the mixed problem for integrable equations and the Cauchy problem is that the scattering matrix here depends implicitly on the time \( t \). To describe the time evolution, we must invoke Eq. (6) and first solve the "scattering problem" for this system conditionally because the coefficient \( u_x(0, t) \) is also unknown. We seek the concrete value of the eigenfunction along the boundary half-line \( x = 0, t > 0 \) from the consistency condition for the scattering problems in \( x \) and \( t \) at the corner point \( x = 0, t = 0 \). Our observation is that for the integrable boundary conditions, the consistency condition allows the efficient evaluation of the eigenfunction along the boundary \( x = 0, t > 0 \).

2. Direct scattering problem on a half-line

We recall that the ISM is based on a nonlocal change of variable that linearizes the initial nonlinear equation (1). The change of variable is defined as a transformation taking the coefficients of system of equations (4) into some “spectral” data that are expressed in terms of the fundamental solution of the system. In accordance with this principle, we construct the matrix solutions of system of equations (4) under the condition that for all real \( \xi = \sqrt{\lambda} \), the solutions have the asymptotic behavior

\[
Y_1(x, t, \xi) \to T(\xi)e^{i\xi \sigma_3}, \quad x \to +\infty, \quad \xi \to +\infty,
\]

\[
Y_2(x, t, \xi) \to T(\xi), \quad x \to 0,
\]

where

\[
\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad T(\xi) = \begin{pmatrix} 1 & 1 \\ i\xi & -i\xi \end{pmatrix}.
\]

As is known [1], the column vectors \( \psi_k(x, t, \xi) \) and \( \phi_k(x, t, \xi) \), \( k = 1, 2 \), of the matrices \( Y_1 = (\psi_1, \phi_1) \) and \( Y_2 = (\phi_2, \psi_2) \) allow an analytic continuation in the parameter \( \xi \) from the real line to the complex plane. In addition, \( Y_2(x, t, \xi) \) is an entire function, and its columns have the following asymptotic behavior in the corresponding half-planes as \( \xi \to \infty \):

\[
\phi_2(x, t, \xi) \to e^{i\xi \xi} \left( \begin{pmatrix} 1 \\ i\xi \end{pmatrix} (1 + O(\xi^{-1})) \right), \quad \text{Im} \xi \leq 0,
\]

\[
\psi_2(x, t, \xi) \to e^{-i\xi \xi} \left( \begin{pmatrix} 1 \\ -i\xi \end{pmatrix} (1 + O(\xi^{-1})) \right), \quad \text{Im} \xi \geq 0.
\]