THE $t \to \infty$ ASYMPTOTIC REGIME OF THE CAUCHY PROBLEM SOLUTION FOR THE TODA CHAIN WITH THRESHOLD-TYPE INITIAL DATA

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We obtain formulas allowing us to find solutions of the Toda chain using the inverse scattering transform method and investigate the $t \to \infty$ asymptotic behavior of the solutions.

1. Introduction

The Toda chain

$$
\begin{align*}
Dt a_n &= \frac{1}{2} a_n (b_n - b_{n+1}), \quad a_n = a_n(t) > 0, \\
Dt b_n &= a_n^2 - a_{n-1}^2, \quad b_n = b_n(t), \quad n \in \mathbb{Z},
\end{align*}
$$

(1.1)

is an interesting example of an integrable differential–difference model with various applications in solid state physics. It is well known that the inverse scattering transform method (ISTM) allows a detailed analysis of the Cauchy problem for the Toda chain in the case with rapidly decreasing initial conditions \cite{1-4} and also in the periodic case \cite{4}. Another interesting class of problems are those with a more complicated behavior of solutions to Eq. (1.1) as $n \to \pm \infty$, where

$$
\begin{align*}
a_n(0) &\to c_1, \quad b_n(0) \to m_1, \quad n \to \infty, \\
\end{align*}
$$

(1.2)

$$
\begin{align*}
a_n(0) &\to c_2, \quad b_n(0) \to m_2, \quad n \to -\infty,
\end{align*}
$$

with $c_1 > 0$ and $c_2 > 0$.

The aim of this work is to construct the $t \to \infty$ asymptotic expansion of the solution to (1.1) and (1.2). Similar problems for the Korteweg–de Vries equations were investigated in \cite{5} and for the Volterra chain in \cite{6}.

In Sec. 2, we analyze the scattering problem for the second-order difference equation whose coefficients tend to different constant limits at $\pm \infty$. A similar problem for the one-dimensional Schrödinger equation was considered in \cite{7}. In the rapidly decreasing case, the scattering problems were studied in \cite{1-4, 8}. In Sec. 3, we use the ISTM to solve the problem in Eqs. (1.1) and (1.2). The last section is devoted to investigating the $t \to \infty$ asymptotic behavior of the solutions of (1.1) and (1.2).

It is assumed in this work that the solution of (1.1) and (1.2) possesses certain smoothness properties and tends to its limits as $n \to \pm \infty$ sufficiently fast and that the auxiliary linear problem has no discrete spectrum.

2. The scattering problem

We consider the difference equation

$$
\begin{align*}
a_{n-1} y_{n-1} + b_n y_n + a_n y_{n+1} &= \lambda y_n, \quad n = 0, \pm 1, \pm 2, \ldots,
\end{align*}
$$

(2.1)
where the coefficients \(a_n\) and \(b_n\) satisfy the conditions
\[
\begin{align*}
  a_n &> 0, \quad \text{Im} \ b_n = 0, \quad n = 0, \pm 1, \ldots, \\
  \sum_{n=1}^{\infty} n(|a_n - c_1| + |b_n - m_1|) + \sum_{n=-\infty}^{1} |n|(|a_n - c_2| + |b_n - m_2|) < \infty
\end{align*}
\] (2.2)
with \(c_1 > 0\) and \(c_2 > 0\).

Let \(\Gamma_j\) denote the complex plane cut along the segment \([m_j - 2c_j, m_j + 2c_j]\) and \(\partial \Gamma_j\) denote its boundary \((j = 1, 2)\). We set
\[
\begin{align*}
  z_j(\lambda) &= \frac{\lambda - m_j}{2c_j} + \sqrt{\frac{(\lambda - m_j)^2}{4c_j^2} - 1},
\end{align*}
\] where the branch of the root in the domain \(\Gamma_j\) is chosen such that \(|z_j(\lambda)| \leq 1\) for \(\lambda \in \Gamma_j\) \((j = 1, 2)\). We often omit the dependence of \(z_j(\lambda)\) on \(\lambda\) in what follows. Thus, in the formulas involving \(z_j\) and \(\lambda\), we always assume that \(z_j\) is as in the above equation.

We let \(f_n(\lambda)\) and \(g_n(\lambda)\) denote the solutions of Eq. (2.1) with the respective asymptotic conditions
\[
\begin{align*}
  \lim_{n \to \infty} f_n(\lambda)z_j^{-n} &= 1, \quad \lambda \in \Gamma_1, \\
  \lim_{n \to \infty} g_n(\lambda)z_j^{-n} &= 1, \quad \lambda \in \Gamma_2.
\end{align*}
\] (2.3)
As long as conditions (2.2) are satisfied, such solutions exist, are unambiguously determined by asymptotic formulas (2.3), and turn out to be regular functions of \(\lambda\) for \(\lambda \in \Gamma_1\) and \(\lambda \in \Gamma_2\) appropriately.

Using the transformation operators, we have the representations
\[
\begin{align*}
  f_n(\lambda) &= a_n z_1^n \left(1 + \sum_{m=1}^{\infty} A_{nm} z_1^m\right), \\
  g_n(\lambda) &= \beta_n z_2^{-n} \left(1 + \sum_{m=-\infty}^{-1} B_{nm} z_2^{-m}\right),
\end{align*}
\] (2.4)
where the quantities \(a_n, A_{nm}, \beta_n, \) and \(B_{nm}\) are related to the coefficients \(a_n\) and \(b_n\) in Eq. (1.1) by
\[
\begin{align*}
  \left(\frac{a_n}{c_1}\right)^2 &= 1 + A_{n-2} - A_{n-1,2} - (A_{n-1} - A_{n-1,1})A_{n1}, \\
  \frac{A_{n+1}}{a_n} &= \frac{a_n}{c_1}, \quad b_n = m_1 + c_1(A_{n1} - A_{n-1,1}),
\end{align*}
\] (2.5)
and
\[
\begin{align*}
  \left(\frac{a_n}{c_2}\right)^2 &= 1 + B_{n+1,-2} - B_{n+2,-2} - (B_{n+1,-1} - B_{n+2,-1})B_{n+1,-1}, \\
  \frac{B_{n+1}}{\beta_n} &= \frac{a_n}{c_2}, \quad b_n = m_2 + c_2(B_{n+1,-1} - B_{n+2,-1}).
\end{align*}
\] (2.6)
We also have the expansions
\[
\begin{align*}
  g_n(\lambda) &= a_1(\lambda)\overline{f_n(\lambda)} + b_1(\lambda)f_n(\lambda), \quad \lambda \in \partial \Gamma_1 \setminus \{m_1 \pm 2c_1\}, \\
  f_n(\lambda) &= a_2(\lambda)\overline{g_n(\lambda)} + b_2(\lambda)g_n(\lambda), \quad \lambda \in \partial \Gamma_2 \setminus \{m_2 \pm 2c_2\}.
\end{align*}
\] (2.7) (2.8)