AN APPROXIMATION DIFFUSION EQUATION
FOR A LONG NARROW CHANNEL WITH VARYING CROSS-SECTIONAL AREA

CLIFFORD S. PATLAK
Theoretical Statistics & Mathematics Section,
National Institute of Mental Health,
Bethesda, Maryland 20014

The diffusion equation for long narrow channels which lie parallel to a rectangular coordinate and have varying cross-sectional areas may be approximated by an equation which involves only one space variable and the average concentration at each value of this space variable. This equation is derived and is discussed along with its assumptions.

Introduction. The passage of particles through a biological system frequently occurs via diffusion of these particles through channels (or pores). In many cases, the channels do not have a uniform cross-sectional area over their entire length. Until now, the only way to treat the problem for such situations has been to solve the general diffusion equation with its three space coordinates (or two if certain symmetries are assumed). If the channels are long and narrow, and if the cross-sectional areas are known as a function of position, it appears plausible that this case could be analyzed approximately by means of an equation involving only one space variable and the average value of the concentration at each value of this space variable. Under certain restrictions this is correct. In this paper such an equation is derived, some applications are discussed and the circumstances under which the equation is valid are explored.

Derivation. The model to be analyzed is as follows: A single channel proceeds along increasing $x$, where $x$ denotes the distance along the rectangular
space coordinate to be considered. The cross-sectional area of the channel, \(A(x)\), is defined as the area of the intersection of the channel with a plane perpendicular to the x-axis. The concentrations of the solute are fixed at the two ends of the channel and the solute diffuses passively within the isotropic channel with a constant diffusion coefficient, \(D\). There is no convective flow of solution within the channel and the surface of the channel is impermeable to solute.

The method used to derive the diffusion equation is the usual conservation of mass approach. Let \(J_x(x, y, z, t)\) be the flux of solute in the x-direction per unit cross-sectional area of channel at the point \((x, y, z)\) at time \(t\). Consider the total space within the channel between the points \(x\) and \((x + dx)\). The number of particles entering this region per unit time across the cross-sectional area at \(x\) is, where \(dA\) is equal to \(dy\,dz\),

\[
\iint_{A(x)} J_x(x, y, z, t) \, dA. \tag{1}
\]

If \(c(x, y, z, t)\) is the concentration of the solute at the point \((x, y, z)\) at time \(t\), then since particles cannot cross the surface of the channel, the net change of the amount of solute within this region is given by

\[
\frac{\partial}{\partial t} \iint_{A(x)} [c(x, y, z, t) \, dA] \, dx
= \iint_{A(x)} J_x(x, y, z, t) \, dA - \iint_{A(x + dx)} J_x(x + dx, y, z, t) \, dA. \tag{2}
\]

Expression (1) is simply a function of \(x\). Thus, the usual Taylor series expansion of the second integral on the right-hand side of (2) may be used and in the limit as \(dx\) goes to 0, (2) reduces to

\[
\frac{\partial}{\partial t} \iint_{A(x)} c(x, y, z, t) \, dA = - \frac{\partial}{\partial x} \iint_{A(x)} J_x(x, y, z, t) \, dA. \tag{3}
\]

Since the diffusion follows Fick's law,

\[
J_x(x, y, z, t) = -D \frac{\partial c(x, y, z, t)}{\partial x}. \tag{4}
\]

Hence, (3) may be written as

\[
\frac{\partial}{\partial t} \iint_{A(x)} c(x, y, z, t) \, dA = D \frac{\partial}{\partial x} \iint_{A(x)} \frac{\partial c(x, y, z, t)}{\partial x} \, dA. \tag{5}
\]