Convolution Operators on $H^p(H^n_k)$

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Abstract Let $G = H^n_k$ be the $(2n + 1)$-dimensional Heisenberg group over local field $K$. In this paper we prove some theorems about convolution operators on $H^p(G)$ and vector-valued Hardy spaces. As an example, the distribution $\int_{S(G)} \varphi dt / |t|$ for some $\varphi \in S(G)$, $\int \varphi = 0$ is a ramified 0-type kernel. These results can be applied to characterize $H^p(G)$ spaces by square functions.

Keywords Convolution operator, Heisenberg group, Local field, Hardy space

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1 Introduction

Let $K$ be a local field and $H^n_k$ be the $(2n + 1)$-dimensional Heisenberg group over $K$. As a manifold, $H^n_k = K \times K^{2n}$. For $(t_i, q_i, p_i) \in H^n_k, t_i \in H^n, i = 1, 2$, a multiplication operation on $H^n_k$ is defined by

$$(t_1, q_1, p_1) \cdot (t_2, q_2, p_2) = (t_1 + t_2 + p_1 q_2, q_1 + q_2, p_1 + p_2).$$

We call $H^n_k$ with this operation a Heisenberg group over local field $K$, and we simply denote it by $G$. A nonarchimedean norm $| \cdot |_H$ on $G$ is defined by

$$|(t, q, p)|_H = \max\{|t|^\frac{1}{2}, |q|, |p|\},$$

where $| \cdot |$ is the nonarchimedean norm on $K$ or $K^n$ (see [7]). For $\lambda \in K^\ast = K \setminus \{0\}$, the dilation $D_\lambda$ defined by $D_\lambda(t, q, p) = (\lambda^{2t}, \lambda q, \lambda p)$ is an automorphism from $G$ onto $G$. If $x \in G, y \in G$, the distance from $x$ to $y$ is $|x^{-1} y|_H$. We can easily verify that $| \cdot |_H$ is a homogeneous norm and $G$ becomes a homogeneous metric space. We simply denote $| \cdot |_H$ by $| \cdot |$, and one can distinguish it with nonarchimedean norms on $K$ and $K^n$ easily.

We now define spheres on $G$. For $k \in \mathbb{Z}/2 = \{0, \pm \frac{1}{2}, \pm \frac{2}{2}, \pm \frac{3}{2}, \cdots \}$, we define $O_0 = \{u \in G : |u| \leq 1\}, O_k = \{u \in G : |u| \leq \gamma^{-k}\}, xO_k = \{u \in G : x^{-1} u \in O_k\}$ where $\gamma = p^c$ for some prime integer $p$ and $c \in \mathbb{N}$. Since Lebesgue measure on $K^{2n+1}$ is the Haar measure on $G$, for $k \in \mathbb{Z}$ the measures of $O_k$ and $O_{k+\frac{1}{2}}$ are $|O_k| = \gamma^{-(k+1)2n-2(k+\frac{1}{2})}$. Since

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We call $Q = 2n + 2$ the homogeneous dimension of $G$. We can easily check that $O_k$ is an open, compact and normal subgroup of $G$. An important fact similar to local fields is: Any two spheres in $G$ are disjoint or one contains the other.

Similarly to the case on local field, the test function class on $G$ is defined as $S(G) = \{ \varphi : \text{supp}\varphi \text{ is compact, } \varphi \text{ is constant on the right and left cosets of some } O_\varphi \text{ in } G \}$. The topology on $S(G)$ is also defined similarly. $S'(G)$ is the space spanned by all continuous linear functionals on $S(G)$ and is endowed with the weak * topology. We call $S'(T)$ the distribution space on $G$. Having done so, we can define right regular functions, Hardy spaces, local Hardy spaces and other type spaces on $G$. We have proved that Hardy spaces on $G$ have many properties similarly to $H^p(K^n)$ such as equivalent characterizations by maximal and grand maximal functions, atomic decompositions. In this paper we will give some special Calderon-Zygmund operators on $H^p(G)$.

These results play an important role in characterizing Hardy spaces by square functions similarly to the case of classical Heisenberg group, but they are much different from the classical case (see [3]).

\section{Convolution Operators on Hardy Spaces}

Suppose $f$ is a measurable function on $G$; its distribution function $\lambda_f : [0, \infty) \to [0, \infty)$ is defined by $\lambda_f(\alpha) = \{|x \in G : |f(x)| > \alpha|\}$, and its nonincreasing rearrangement $f^* : [0, \infty) \to [0, \infty]$ is defined by $f^*(t) = \inf\{\alpha : \lambda_f(\alpha) \leq t\}$. As is well known (see [6] or [3]),

$$\int_G |f(x)|^p \, dx = -\int_0^\infty \alpha^{p-1} \lambda_f(\alpha) d\alpha = \int_0^\infty |f^*(t)|^p \, dt.$$

For $0 < p < \infty$, weak $L^p$ is the space of functions $f$ such that $[f]^p = \sup_{\alpha > 0} \alpha^p \lambda_f(\alpha) = \sup_{t > 0} t^{1/p} f^*(t) < \infty$. By the Chebyshev's inequality it is easily checked that $[f]^p \leq \|f\|_p$. $[\cdot]$ is not a norm, but it defines a topology on weak $L^p$. A subadditive operator which is bounded from $L^p$ to weak $L^q$ is said to be of weak type $(p, q)$.

Just like the classical case, we have the generalized Young's inequality.

**Proposition 2.1** Suppose $1 < p < \infty, 1 < q, r < \infty$ and $1/p + 1/q = 1/r + 1$. If $f \in L^p$ and $g \in \text{weak } L^q$, then $f \ast g \in \text{weak } L^r$, and there exists $c_1 = c_1(p, q)$ such that $\|f \ast g\|_r \leq c_1 \|f\|_p \|g\|_q$.

Moreover, if $p > 1$, then $f \ast g \in L^r$ and there exists $c_2 = c_2(p, q)$ such that $\|f \ast g\|_r \leq c_2 \|f\|_p \|g\|_q$.

A kernel of $\alpha$-type for $0 < \alpha < Q$ is a function $K$ on $G$ which is continuous on $G \setminus \{0\}$ and satisfies

$$|K(x)| \leq A|x|^{\alpha - Q}, \quad x \neq 0. \quad (2.1)$$

The above estimate implies that $K$ is locally integrable and bounded near infinity, so that $K$ defines a distribution.

**Proposition 2.2** Suppose $0 < \alpha < Q, 1 < p < Q/\alpha$ and $1/q = 1 - \alpha/Q$. If $K$ satisfies (2.1), then the operator $T_K : f \mapsto f \ast K$ is bounded from $L^p$ to $L^q$. Also $T_K$ is of weak type $(1, Q/(Q - \alpha))$.

**Proof** The estimate $|K(x)| \leq A|x|^{\alpha - Q}$ implies that $K \in \text{weak } L^{Q/(Q - \alpha)}$, and so the results follow from Proposition 2.1.

The more interesting and more subtle is the limiting case $\alpha = 0$. In this case (2.1) neither implies the local integrability of $K$ (and hence that $K$ defines a distribution), nor yields automatically $L^p$ boundedness theorem. Hence we assume these conditions separately as follows.