A Discrete Singular Integral Operator
(In Memory of Professor Long Ruilin)

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Abstract Suppose that \( \{\alpha_k\}_{k=-\infty}^{\infty} \) is a Lacunary sequence of positive numbers satisfying \( \inf_k \alpha_{k+1}/\alpha_k = \alpha > 1 \) and that \( \Omega(y') \) is a function in the Besov space \( B_{1,1}^{0,1}(S^{n-1}) \) where \( S^{n-1} \) is the unit sphere on \( \mathbb{R}^n (n \geq 2) \). We prove that if \( \int_{S^{n-1}} \Omega(y')d\sigma(y') = 0 \) then the discrete singular integral operator

\[
T_\Omega f(x) = \sum_{k=-\infty}^{\infty} \int_{S^{n-1}} f(x - \alpha_k y')\Omega(y')d\sigma(y')
\]

and the associated maximal operator

\[
T_\Omega^* f(x) = \sup_N \left| \sum_{k=N}^{\infty} \int_{S^{n-1}} f(x - \alpha_k y')\Omega(y')d\sigma(y') \right|
\]

are both bounded in the space \( L^2(\mathbb{R}^n) \).

The theorems in this paper improve a result by Duoandikoetxea and Rubio de Francia[1] in the \( L^2 \) case.

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1 Introduction

Let $y'$ be a point on the unit sphere $S^{n-1}$ on $\mathbb{R}^n (n \geq 2)$ and $d\sigma(y')$ be the induced Lebesgue measure on $S^{n-1}$. In [1], Duoandikoetxea and Rubio de Francia considered the discrete singular integral

\[ T_n(f)(x) = \sum_{k \in \mathbb{Z}} \int_{S^{n-1}} f(x - 2^k y') \Omega(y') d\sigma(y'), \quad (1.1) \]

where $\Omega(y')$ is a function in the Sobolev space $L^1_\beta(S^{n-1})$ for some $\beta > 0$, and proved

\[ \|T_n(f)\|_p \leq C \|f\|_p \quad \text{for any} \quad p \in (1, \infty). \quad (1.2) \]

The main purpose of this article is to prove the above boundedness property under a weaker condition on $\Omega(y')$ in the case of $p = 2$. We let \{\alpha_k\}_{k=-\infty}^{\infty} be a Lacunar \gamma sequence of positive numbers satisfying $\inf_{k} \alpha_{k+1}/\alpha_k = \gamma > 1$ and let $\Omega(y')$ be a suitable function on $S^{n-1}$. The truncated discrete singular integral $T_{n,N}f(x)$ is defined by

\[ T_{n,N}f(x) = \sum_{k \geq N} \int_{S^{n-1}} f(x - \alpha_k y') \Omega(y') d\sigma(y') \quad (1.3) \]

and the maximal operator $T_n^\gamma$ is defined by

\[ T_n^\gamma f(x) = \sup_{N \in \mathbb{Z}} |T_{n,N}f(x)|. \]

We have the following theorem.

Theorem 1 If $\Omega \in B^{0,1}_1(S^{n-1})$ satisfying $\int_{S^{n-1}} \Omega(y') d\sigma(y') = 0$, then

(i) $\|T_{n,N}f\|_{L^2(\mathbb{R}^n)} \leq C \|f\|_{L^2(\mathbb{R}^n)} \|\Omega\|_{B^{0,1}_1(S^{n-1})}$, 
(ii) $\|T_n^\gamma f\|_{L^2(\mathbb{R}^n)} \leq C \|f\|_{L^2(\mathbb{R}^n)} \|\Omega\|_{B^{0,1}_1(S^{n-1})}$,

where $C$ is a constant independent of $f$, $N$ and $\Omega$.

Remark Duoandikoetxea and Rubio de Francia obtained the $L^p$ boundedness of $T_{n,N}$ under the assumption $\Omega \in L^1_\beta(S^{n-1})$ for some $\beta > 0$. Here we point out that

\[ \bigcup_{\beta > 0} L^1_\beta(S^{n-1}) \subset B^{0,1}_1(S^{n-1}) \quad (1.4) \]

and the inclusion is proper. To see this, we consider the inhomogeneous Triebel-Lizorkin spaces $F^{\alpha,q}_p$ and inhomogeneous Besov spaces $B^{\alpha,q}_p$ on $S^{n-1}$ which are defined by using a standard method (see [2]). By the compactness of $S^{n-1}$ we have

\[ B^{0,1}_1(S^{n-1}) \supset \bigcup_{p>1, \alpha > 0} B^{\alpha,2}_p(S^{n-1}) \quad (1.5) \]

and the inclusion is proper. Since for any $p > 1$ and $\alpha > 0$, $B^{\alpha,2}_p \supset F^{\alpha,2}_p = L^p_\alpha$, we obtain

\[ \bigcup_{p>1, \alpha > 0} L^p_\alpha(S^{n-1}) \subset B^{0,1}_1(S^{n-1}). \quad (1.6) \]

By the Sobolev imbedding theory, (1.4) follows from (1.6).

This paper is organized as follows. In the second section we will give the precise definition of the Besov space $B^{0,1}_1(S^{n-1})$, as well as some lemmas which will be used to prove the main theorem. Theorem 1 will be proved in section 3. In the fourth section, we will give an extension of Theorem 1 that is related to polynomial phases.