The Existence of Minimal Honest Polynomial Degree Below and Recursively Enumerable Degrees

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Abstract. In [1] Homer introduced the honest polynomial reducibility and proved that under this new reducibility a set of minimal degree below $0^*$ is constructed under the assumption that $P = NP$. In this paper we will prove that under the same assumption a set of minimal degree can be constructed below any recursively enumerable degrees. So under the honest polynomial reducibility a set of low minimal degree does exist.

In [1] Homer introduced the honest polynomial reducibility and proved that under this new reducibility a set of minimal degree below $0^*$ is constructed under the assumption that $P = NP$. In this paper we will prove that under the same assumption a set of minimal degree can be constructed below any recursively enumerable degrees. So under the honest polynomial reducibility a set of low minimal degree does exist.

We consider computations of oracle Turing machines. Without loss of generality the tape alphabet of all oracle Turing machines is $\Sigma = \{0, 1\}$ and all languages will be subset of $\Sigma^*$. For $x \in \Sigma^*$, $|x|$ denotes the length of $x$. We use $\lambda$ to denote the empty word.

For any oracle Turing machine $T$ and set $S$ let $T^S$ denote machine $T$ with oracle $S$. Oracle Turing machines run in polynomial time if there is a polynomial $p$ such that for any oracle set and any input of length $n$ the machine halts in $p(n)$ steps. A set $A$ is Turing reducible to $B$ in polynomial time ($A \leq_T B$) if there is a polynomial time oracle Turing machine $T$ with oracle $B$ such that for all $x \in \Sigma^*$, $x \in A$ iff $T^B$ accepts $x$. The oracle machine $T$ is sometimes referred to as a reduction procedure as it reduces $A$ to $B$.

A is honest Turing reducible to $B$ ($A \leq_{h} B$) if there is an oracle Turing machine $M$ such that

1. $A \leq_{h} B$ via oracle Turing machine $M$, and
2. there is a polynomial $q$ such that for all $x$, if $M^B(x)$ queries oracle $B$ about a string $y$ then $q(|y|) \geq |x|$.

We can define the equivalence relation $\equiv_{h}$ by $A \equiv_{h} B$ iff $A \leq_{h} B$ and $B \leq_{h} A$. The equivalence classes of this relation are called honest Turing degrees.

We say that a set $C$ is minimal with respect to $\leq_{h}$ if (1) $C \notin P$. (2) for any set $D$ if $D \leq_{h} C$ and $C \leq_{h} D$ then $D \in P$. A minimal degree for $\leq_{h}$ is one which is made up of minimal sets.

Let $\{P_i\}$, $i = 0, 1, 2, \ldots$, be the enumeration of all polynomial time sets.

Consider an enumeration $\{M_i\}$, $i = 1, 2, \ldots$, of polynomial time bounded oracle Turing machines. Say that $M_i$ runs in polynomial time $p_i$. For $M_i$ we have a polynomial $q_i(x) = x'$. We define a $\leq_{h}$ reduction procedure by, for input $x$, $M_i$ carries out its $p_i$ time bounded computation for input $x$ with the added constraint that whenever $M_i$ on input $x$ queries its oracle about a string $y$, the
procedure first checks if $q_{a}(b) \geq |a|$. If it is, the computation proceeds as usual; if not, the computation halts and rejects $x$. Let $L(M_{i}, A)$ be the language accepted by $M_{i}$ using oracle set $A$.

By a string here we mean an element of $\Sigma^{*}$. Concatenation of strings is indicated by juxtaposition. Let $a(k) = \text{the } (k + 1) \text{ th bit of } a$, so $a = a(0) \ldots a(|a| - 1)$. Two strings $a$ and $\beta$ are incompatible if $\exists k < |a| \land k < |\beta| \land a(k) \neq \beta(k)$. Otherwise they are said to be compatible. $\alpha$ extends $\beta (\beta \subset a)$, if $a$ and $\beta$ are compatible and $|\beta| < |a|$. We write $\beta \preceq a$ to mean $\alpha$ extends $\beta$ or they are equal.

A tree is a partial function $T$ from $\Sigma^{*}$ to $\Sigma^{*}$ satisfying

1. $\forall a \text{ if } T(a0)$ and $T(a1)$ are defined then they extend $T(a)$ and are incompatible.
2. $\forall a \text{ if } T(a0)$ is defined then $T(a0)$ extends $T(a)$.
3. $\forall a, \beta \text{ if } T(a)$ is defined and $\beta \subset a$ then $T(\beta)$ is defined.

A node $\xi$ of $T$ is an element of the range of $T$. The trees we construct will have the property that for every node $T(a)$ either both $T(a0)$ and $T(a1)$ will be defined or only $T(a0)$ will be defined. $T(a0)$ and $T(a1)$ are called the immediate extensions of $T(a)$.

A string $a$ is on $T(a \preceq T)$ iff it is a node of $T$: A tree $T'$ is a subtree of tree $T$ if every string on $T'$ is on $T$. Let $\alpha$ be a string and $X$ be a tally set (that is $X \preceq \{1\}^{*}$). We write $a \preceq X$ to mean $\forall k < |a| (a(k) = 1 \text{ iff } P^{1} \_k \in X)$. We say that a tally set $X$ is on $T$ if $a \preceq X$ for infinitely many $a$ on $T$.

A tree $T$ is uniform if $\forall a_{0}, a_{1}$ on $T \forall \beta$ on $T$ if $|a_{0}| = |\alpha_{1}| \land \beta$ is an extension of $a_{0}$, then $\alpha_{1}\beta(|a_{0}|) \ldots \beta(|\beta| - 1)$ is the extension of $\alpha_{1}$ on $T$. Intuitively $T$ is uniform if for $a_{0}, a_{1}$ on $T$ of the same length, they both have the same number of extensions and these extensions agree on all bits past $|a_{0}| - 1$.

A tree $T$ has close extensions if there is a polynomial $b$ such that $\forall a \text{ on } T$ (the length of immediate extension(s) on $T$ of $a$ are bounded above by $b(|a|)$).

A tree $T$ is called acceptable if $T$ is uniform, $T$ has close extensions, the relation "$\alpha$ is on $T$" is in $P$, and every node of $T$ has two incompatible extensions on $T$.

Let $C$ be a non-recursive recursively enumerable set and let $k: N \rightarrow N$ be a one-one recursive function enumerating $C$. Let $C^{s} = \{k(x) \mid x < s\}$.

We will construct a set $A \subseteq r C$ by recursive approximation. Thus we define a recursive sequence of strings $\{a_{s} \mid s \in N\}$ and let $A(x) = \lim_{s} a_{s}(x)$ for all $x \in N$. We will subject the sequence $\{a_{s} \mid s \in N\}$ to the following constraint

(4) $x, s \in N|C^{s}| x = C^{s}\downarrow x \rightarrow a_{s}(x) \preceq A$.

According to the Yates permitting lemma, (4) will guarantee that $A \subseteq r C$.

Besides (4) the set $A$ should meet the following requirements for all $e$

$S_{e}$: $A \neq P_{e}$, $P_{e}$ is the $(e + 1)$th polynomial time set.

$R_{e}$: Either $I(M_{e}, A) \in P$ or $A \preceq I(M_{e}, A)$.

Let $T, T^{*}$ be finite uniform trees and have close extension such that $T^{*} \preceq T$. Let $e, s \in N$ and string $\alpha$ be given. Define the tree $T^{*} = PSp(T, T^{*}, \alpha, e, s)$ as follows.

$$
T^{*}(\xi) = \begin{cases} 
T^{*}(\xi) & \text{if } T^{*}(\xi) \downarrow, \\
T(\lambda) & \text{if } T(\lambda) \downarrow \land T^{*}(\xi) \uparrow \land \xi = \lambda, \\
\uparrow & \text{if } T(\lambda) \uparrow \land \xi = \lambda.
\end{cases}
$$