A Bayesian Approach to the Multivariate Behrens-Fisher Problem Under the Assumption of Proportional Covariance Matrices

D. G. NEL and P. C. N. GROENEWALD

Dept. Mathematical Statistics, University of the Orange Free State
339-Bloemfontein, 9300-South Africa

SUMMARY

Two independent random samples of sizes $N_1$ and $N_2$ from multivariate normal populations $N_p(\theta_1, \Sigma_1)$ and $N_p(\theta_2, \Sigma_2)$ are considered. Under the null hypothesis $H_0: \theta_1 = \theta_2$, a single $\theta$ is generated from a $N_p(\mu, \Sigma)$ prior distribution, while under $H_1: \theta_1 \neq \theta_2$ two means are generated from the exchangeable prior $N_p(\mu, \sigma)$. In both cases $\Sigma$ will be assumed to have a vague prior distribution. For a simple covariance structure, the Bayes factor $B$ and minimum Bayes factor in favour of the null hypotheses is derived. The Bayes risk for each hypothesis is derived and a strategy is discussed for using the Bayes factor and Bayes risks to test the hypothesis.

Keywords: BAYES RISK; HYPOTHESIS TESTING; POSTERIOR PROBABILITY.

1. INTRODUCTION

The well known Behrens-Fisher problem (Behrens (1929), Fisher (1939)) deals with the distribution of a test statistic under the assumption of equality of two normal means, but unequal and unknown covariance matrices. There is a vast literature devoted to this problem within classical statistics with a number of approximate solutions and some exact results. For a summary of these results, see Nel, et al. (1990).
A Bayesian solution in the univariate case is illustrated by Box and Tiao (1973, pp. 107-110) where the exact posterior density of the difference between two normal means is given. Earlier an approximate solution was proposed by Patil (1964). Broemeling et al. (1990) derived an approximation to the posterior distribution of a set of contrasts among k normal means.

All of the above approaches aim at the posterior density function of the difference between the two normal means under noninformative priors. Acceptance or rejection of the hypothesis of equality of means is then based on a HPD region or credibility ellipsoid.

Our purpose with this paper is twofold: Firstly to derive posterior results for the "sharp" null hypothesis of equality of means. Although some may argue that sharp hypotheses are unrealistic, they are still points of special interest, and can be regarded as approximations to some small interval as argued by Jeffreys (1961) and Zellner (1984). This allows a more direct comparison between Bayesian posterior probability and the p-value of classical testing which are too often given a Bayesian interpretation by nonstatisticians.

Secondly, in the spirit of Dickey (1973) on scientific reporting and personal probabilities, posterior results are derived over a class of prior distributions and a strategy for reaching a decision is proposed, based on a personal loss function.

So we will consider two random samples $X_{i1}, \ldots, X_{iN_i}$, $i = 1, 2$ of $p$-component vectors from a multivariate normal distribution $X_{ij} \mid \theta_i, \Sigma_i \sim N_p(\theta_i, \Sigma_i)$, $i = 1, 2$; $j = 1, \ldots, N_i$.

We are testing the hypothesis $H_0 : \theta_1 = \theta_2$ against the alternative $H_1 : \theta_1 \neq \theta_2$. Under $H_1$ we assume that the mean vectors $\theta_i; i = 1, 2$ can be regarded as having been generated by an exchangeable prior distribution $N_p(\mu, \Sigma)$. Under the null hypothesis the assumption is that a single $\theta$ was generated by the $N_p(\mu, \Sigma)$ distribution.

We also assume that $\mu$ and $\Sigma$ have a vague prior density function $g(\mu, \Sigma) \propto |\Sigma|^{-\frac{1}{2}(p+1)}$, irrespective of which hypothesis is true. The joint prior density function of $\theta_i, i = 1, 2$, $\mu$ and $\Sigma$ can be presented as: