Zeta-Functions of Ideal Classes in Quadratic Fields
and their Zeros on the Critical Line

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§ 1. If $K$ is a quadratic field, and $\mathfrak{C}$ is an ideal class in $K$, the Dedekind Zeta-function of the class $\mathfrak{C}$ in $K$ is defined by the Dirichlet series

$$\zeta_K(s, \mathfrak{C}) = \sum_{a \in \mathfrak{C}} \frac{1}{(Na)^s},$$

(1.1)

where $s$ is a complex variable, $s = \sigma + it$, $\sigma > 1$; the sum extends over the non-zero integral ideals $a$ in $\mathfrak{C}$, and $Na$ is the norm of $a$. The function $\zeta(s, \mathfrak{C})$ satisfies a functional equation, the form of which depends on the nature of $K$. If $K$ is an imaginary quadratic field, say $K = \mathbb{Q}(\sqrt{-d})$, $d > 0$, then we have

$$\left(\frac{\sqrt{d}}{2 \pi}\right)^s \Gamma(s) \zeta(s, \mathfrak{C}) = \left(\frac{\sqrt{d}}{2 \pi}\right)^{1-s} \Gamma(1-s) \zeta(1-s, \mathfrak{C}).$$

(1.2)

If $K$ is a real quadratic field, say $K = \mathbb{Q}(\sqrt{d})$, then the corresponding functional equation has a different gamma-factor, and is of the form

$$\left(\frac{\sqrt{d}}{\pi}\right)^s \Gamma^2\left(\frac{s}{2}\right) \zeta(s, \mathfrak{C}) = \left(\frac{\sqrt{d}}{\pi}\right)^{1-s} \Gamma^2\left(\frac{1-s}{2}\right) \zeta(1-s, \mathfrak{C}).$$

(1.3)

The equations (1.2) and (1.3) take this form since the field is quadratic, so that $\zeta(s, \mathfrak{C}) = \zeta(s, \overline{\mathfrak{C}})$, where $\overline{\mathfrak{C}}$ is the class conjugate to $\mathfrak{C}$. It is known, after HECKE [1], that the Zeta-function of an ideal class in an imaginary quadratic field has an infinity of zeros on the critical line. It is not known, however, whether the corresponding result is true in the case of a real quadratic field. The Dirichlet series for $\zeta(s, \mathfrak{C})$, in both cases, can be written in the form

$$\zeta_K(s, \mathfrak{C}) = \sum_{m=1}^{\infty} \frac{a_m}{m^s}, \quad \sigma > 1,$$

(1.4)

where

$$a_m = \sum_{a \in \mathfrak{C}} 1,$$

and it is known, after Dedekind, that

$$A(x) = \sum_{m \leq x} a_m \sim \kappa x, \quad 0 < \kappa < \infty.$$
Our object is to give, in both cases, a sufficient condition, in terms of an estimate on $a_m$, for the existence of an infinity of zeros on the critical line, and to show that that estimate is actually true. Corresponding estimates exist for fields of degree $n > 2$, and we postpone the more general problem to a later occasion.

What we actually require is that

$$\sum_{m \leq T} a_m e^{2\pi i mx} = o(T), \quad \text{as} \quad T \to \infty,$$

for any irrational $x$. This is obtained from Hermann Weyl's estimate of exponential sums. The connection between this estimate, and the existence of an infinity of zeros, on the critical line, for the corresponding Zeta-function, is established here by a combination of van der Corput's method [3, Ch. IV] for estimating exponential integrals, with the Hardy-Littlewood proof [3, p. 219] of Hardy's theorem establishing the existence of an infinity of zeros of the classical Riemann Zeta-function on the critical line. We prove the following simple results.

**Theorem 1.** For every irrational number $x$, we have the estimate

$$\sum_{m \leq T} a_m e^{2\pi i mx} = o(T), \quad \text{as} \quad T \to \infty. \quad (1.6)$$

**Theorem 2.** The function $\zeta_K(\frac{1}{2} + it, \mathfrak{C})$ vanishes for an infinity of real values of $t$.

§ 2. **Proof of Theorem 1.** We consider two cases, according as the given field $K$ is real or imaginary.

**Case (i).** Let $K = \mathbb{Q}(\sqrt{d}), d > 0$. From the definition of $a_m$ it is seen that (refer, for example [5, p. 87]) if

$$S(T) = \sum_{m \leq T} a_m e^{2\pi i mx},$$

then

$$2S(T) = \sum e^{2\pi i [P(k, l)](N(b))^{-1}x}. \quad (2.1)$$

Here $b$ is a non-zero integral ideal in the class $\mathfrak{C}^{-1}$ (where $\mathfrak{C}$ is the given ideal class) with a base $(a, b)$, and if $a', b'$ denote the conjugates of $a$ and $b$, then

$$P(k, l) = (ka + lb)(ka' + lb').$$

The summation in (2.1) is over integers $k$ and $l$, such that

$$|P(k, l)| \leq T(N b), \quad 1 \leq \frac{|ka + lb|}{|ka' + lb'|} < \eta^2. \quad (2.2)$$