THE INTERSECTION THEOREM FOR ORDERINGS OF HIGHER LEVEL IN RINGS

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This short note is meant as a supplement to the paper "On Rings admitting Orderings and 2-primary Orderings of Higher Level" by E. Becker and D. Gondard ([4]), where an intersection theorem for 2-primary orderings of higher level has been proved ([4], Proposition 2.6). We will show that the same characterization holds for orderings of arbitrary level. This result finds several applications. For example, it is useful for the continuous representation of sums of 2n-th powers in function fields (see [8]) and it can be applied to derive several Null- and Positivstellensätze for generalized real closed fields (see [5]). As a further example we will prove a strict "Positivstellensatz of higher level" for a certain class of formally real fields. For unexplained notions we refer the reader to [4].

First we will state some results about modules over preorderings. Let $A$ be a commutative ring with 1 and let $T \subset A$ be a fixed preordering of level $n$. A subset $M \subset A$ is called a $T$-module if

$$M + M \subset M, T \cdot M \subset M, 1 \in M, -1 \notin M.$$ 

For an ideal $I \subset A$ we denote by $M/I \subset A/I$ the image of $M$ with respect to the canonical projection $A \to A/I$. Note that $M/I$ is a $T/I$-module if and only if $1 + M \cap I = \emptyset$, i.e. $M + I$ is a $T$-module. Finally, we call $I$ $M$-convex if for all $m, m' \in M$ we have

$$m + m' \in I \Rightarrow m, m' \in I.$$ 

We have the following characterization of $M$-convex ideals:

**Lemma 1:** Let $M$ be a $T$-module and $I \subset A$ a proper ideal. Then the following statements are equivalent:

1. $I$ is $M$-convex.
2. $M' := M + I$ is a $T$-module with $M' \cap -M' = I$.
3. $M/I$ is a $T/I$-module with $M/I \cap -M/I = (0)$.

**Proof:** The equivalence of (2) and (3) is obvious. Thus it remains to prove (1) $\Leftrightarrow$ (2): First assume $I$ is $M$-convex. Then $1 + M \cap I = \emptyset$, as $I$ is proper. Hence $-1 \notin M'$ showing that $M'$ is a $T$-module. Obviously we have $I \subset M' \cap -M'$. So let $a \in M' \cap -M'$. Choose $m_1, m_2 \in M, b_1, b_2 \in I$ with $a = m_1 + b_1 = -m_2 + b_2$. Then

$$a + m_2 - b_1 = m_1 + m_2 \in I.$$

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Hence $m_1, m_2 \in I$, which implies $a \in I$. Conversely, assume that $M$ and $I$ satisfy (2). Let $m_1, m_2 \in M$ with $m_1 + m_2 \in I$. Then $-m_1 - m_2 \in M'$. Thus we have $-m_1, -m_2 \in M' \cap -M' = I$. \[ \square \]

If $M$ is a $T$-module, then $M \cap -M$ need not be an ideal. However, we have (cf. [7], Proposition 1.3):

**Lemma 2:** Let $M$ be a $T$-module and let $I \subseteq A$ be the ideal generated by $M \cap -M$. Then $I$ is a proper $M$-convex ideal.

**Proof:** First note $S := (T - T)(M \cap -M) \subseteq -M$. Moreover, $S$ is additively closed. Now let $a = \sum a_i m_i \in I$, with $a_i \in A, m_i \in M \cap -M$. By [7], Proposition 1.3, there exists $k \in \mathbb{N}$ such that $ka_i \in T - T$ for all $i$. Hence $ka \in S \subseteq -M$. In particular, $1 \not\in I$, as $k \not\in -M$ for all $k \in \mathbb{N}$. Thus $I$ is a proper ideal. Now let $m_1, m_2 \in M$ with $m_1 + m_2 \in I$. Then there exists $k \in \mathbb{N}$ with $k(m_1 + m_2) \in -M$. Hence $-m_1, -m_2 \in M$ which shows $m_1, m_2 \in I$. Thus, $I$ is a proper $M$-convex ideal. \[ \square \]

Our next result also can be deduced from the proof of [7], Theorem 1.6. However, we will give an additional proof being closer to the well known argument in the case that $M$ is a preordering.

**Proposition 3:** Let $M$ be a $T$-module and $\varphi \subseteq A$ an ideal which is maximal with respect to the property that $1 + M \cap \varphi = 0$. Then $\varphi$ is a $M$-convex prime ideal.

**Proof:** By assumption, $M' := M + \varphi$ is a $T$-module. Let $\varphi'$ be the ideal generated by $M' \cap -M'$. Then $1 + M' \cap \varphi' = \emptyset$, by the last result. In particular, $1 + M \cap \varphi' = \emptyset$. Thus $\varphi = \varphi'$, as $\varphi$ is maximal. Hence $\varphi = M' \cap -M'$ which implies that $\varphi$ is $M'$-convex, by Lemma 1. But then $\varphi$ is $M$-convex as well. Next assume $ab \in \varphi$ for some $a, b \in A$. If $b \notin \varphi$ we find $m \in M$ and $c \in A$ such that $1 + m \in cb + \varphi$, by the maximality of $\varphi$. But then

$$a^{2n} + a^{2n}m \in cb^{2n} + \varphi \subseteq \varphi.$$ 

Hence $a^{2n} \in \varphi$, as $\varphi$ is $M$-convex. Thus it remains to show that $\varphi$ is radically closed. By passing over from $A$ resp. $M$ to $A/\varphi$ resp. $M/\varphi$ we may assume w.l.o.g. that $\varphi = (0)$. Assume by way of contradiction that $A$ is not reduced. Then we may argue as in the proof of [7], Theorem 1.6. Namely, choose $0 \neq a \in \text{Nil}(A)$ with $a^2 = 0$. Then there exist $b \in A$ and $m \in M$ with $ab = 1 + m$. Since $(1 + m)^2 = 0$, we get

$$(1 - (1 + m))^{2n} = 1 - 2n - 2nm \in M,$$

which implies the contradiction $-1 \in M$. \[ \square \]