ON BERTINI’S THEOREM

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We study the singularities of a generic element of a linear system of divisors on a smooth projective variety inside the base locus of the linear system, and give a generalization of Bertini’s theorem.

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Let $X$ be a smooth projective variety with $\dim X \geq 2$. If

$$|L| = \{\lambda_0 Q_0 + \lambda_1 Q_1 + \cdots + \lambda_k Q_k | (\lambda_0, \cdots, \lambda_k) \in \mathbb{P}^k\}$$

is a $k$-dimensional linear system of divisors without fixed component on $X$, then Bertini’s theorem [B] says that; (1). The generic element $D \in |L|$ is non-singular away from the base locus $B$ of $|L|$. (2). If $P \in X$ is an arbitrary point on $X$ and $D \in |L|$ is a generic element, then

$$\inf_{i \in \{0, 1, \cdots, k\}} \text{mult}_P Q_i \geq \text{mult}_P D - 1,$$

in particular, if $P \notin B$ the base locus of $|L|$, that is, $P \notin Q_i$ for some $i$, then we must have $\text{mult}_P D \leq 1$, that is, the generic element $D$ is smooth at any point $P$ outside of the base locus $B$ of $|L|$. Therefore, one can view statement (1) in Bertini’s theorem as a special case of (2).

In this paper, we study the question of what type of singularities precisely a generic element of a linear system can afford inside the base locus, and give a generalization of statement (2) in Bertini’s theorem.

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In order to do that, we need to first fix some notations. If \( D \in |L| \) is a reduced and irreducible divisor on \( X \), then in general \( D \) is singular. According to Hironaka [H], we have a desingularization of \( D \):

\[ X_{m+1} \xrightarrow{\pi^{m+1}} X_m \xrightarrow{\pi^m} \cdots \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} X_0 = X, \]

so that the proper transform \( \tilde{D} \) of \( D \) in \( X_{m+1} \) is smooth. Here \( X_j \xrightarrow{\pi_j} X_{j-1} \) is the blow-up of \( X_{j-1} \) along a \( \nu_{j-1} \)-dimensional submanifold \( Y_{j-1} \) with \( E_{j-1} \subset X_j \) the exceptional divisor. If \( Y_{j-1} \) is a \( \mu_{j-1} \)-fold singular submanifold of the proper transform of \( D \) in \( X_{j-1} \), then we say that \( D \) has a type \( \mu = (\mu_j, Y_j, E_j \mid j \in \{0, 1, \ldots, m\}) \) singularity.

Now if \( Z \subset X \) is another divisor, such that

\[ \pi_j^* \left( \cdots (\pi_2^*(\pi_1^*(Z) - \delta_0 E_0) - \delta_1 E_1) - \cdots \right) - \delta_{j-1} E_{j-1} \]

is an effective divisor for \( j = 1, 2, \ldots, m + 1 \), then we say that \( Z \) has a weak type \( \delta = (\delta_j, Y_j, E_j \mid j \in \{0, 1, \ldots, m\}) \) singularity.

We now state our results, which can be viewed as a generalization of Bertini’s theorem.

**Theorem.** Assume that

\[ |L| = \{\lambda_0 Q_0 + \lambda_1 Q_1 + \cdots + \lambda_k Q_k \mid (\lambda_0, \cdots, \lambda_k) \in \mathbb{P}^k\} \]

is a linear system of divisors without fixed component on a smooth variety \( X \) with \( \dim X \geq 2 \), and that the generic element of \( |L| \) is irreducible. If a generic divisor \( D_\lambda \in |L| \) has a type \( \mu(\lambda) = (\mu_j, Y_j(\lambda), E_j(\lambda) \mid j \in \{0, 1, \ldots, m\}) \) singularity inside the base locus \( B \), then every base element \( Q_i \ (i = 0, 1, \cdots, k) \) of the linear system has a weak type \( \mu(\lambda) - 1 = (\mu_j - 1, Y_j(\lambda), E_j(\lambda) \mid j \in \{0, 1, \ldots, m\}) \) singularity. Furthermore, if \( Y_0(\lambda) = Y_0 \) does not move, then \( Q_i \) has multiplicity \( \mu_0 \) along \( Y_0 \).

As a special case of the above theorem, one can immediately recover the classical Bertini’s theorem we mentioned at the beginning of this paper.

If \( |L| \) is a linear system of divisors of dimension \( k \) on \( X \) with \( k \geq 1 \), and \( f \) is the rational map from \( X \) to \( \mathbb{P}^k \) associated to \( |L| \), then by Bertini’s theorem [FL], we know that the generic element of \( |L| \) is irreducible if \( \dim f(X) \geq 2 \). Moreover, \( |L| \) is composed with a pencil if \( \dim f(X) = 1 \).

The method we will use to prove the theorem is deformation of singularities as we did in [X1] and [X2].

Throughout this paper we work over the field of complex number \( \mathbb{C} \).

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We now start the proof of our theorem.