Logarithmic Density and Measures on Semigroups

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Davenport and Erdős [3] proved that every set $A$ of integers with the
property that $a \in A$ implies $an \in A$ for all $n$ (multiplicative ideal) has a
logarithmic density. I generalized [5] this result to sets with the property that
if for some numbers $a, b, n$ we have $a \in A, b \in A$ and $an \in A$, then necessarily
$bn \in A$, which I call quasi-ideals.

Here a new proof of this theorem is given, applying a result on convolution
of measures on discrete semigroups. This leads to further generalizations,
including an improvement of a result of Warlimont [8] on ideals in abstract
arithmetic semigroups.

1. Introduction

The asymptotic density of a set $A$ of natural numbers is defined by

$$d(A) = \lim_{N \to \infty} \frac{1}{N} \sum_{a \in A, a \leq N} 1$$

if this limit exists, and its logarithmic density is

$$\lambda(A) = \lim_{N \to \infty} \frac{1}{\log N} \sum_{a \in A, a \leq N} 1/a.$$  (1.1)

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It is well known that each set which has an asymptotic density has also a logarithmic one (and the values are equal), but a set may have a logarithmic density without having an asymptotic one.

In 1936 Davenport and Erdős [3] proved that every multiplicative ideal of the set $\mathbb{N}$ of positive integers (that is, every set $A$ with the property that $a \in A$ implies $an \in A$ for all $n \in \mathbb{N}$) has a logarithmic density. Besicovitch [2] constructed a multiplicative ideal that has no asymptotic density.

I generalized this result as follows [5] (the fact that this is a generalization will be seen from Statement 1.1 below).

**Theorem 1.** Let $f$ be a homomorphism from the multiplicative semigroup of integers to a commutative semigroup $H$ (or, in other words, an $H$-valued completely multiplicative function). Then for every $g \in H$ the set

$$A_g = \{n \in \mathbb{N} : f(n) = g\}$$

possesses a logarithmic density.

In [4] I gave the following inner characterization of the possible sets $A_g$.

**Statement 1.1.** For a set $A \subseteq \mathbb{N}$ the following are equivalent:

a) there is a semigroup $H$, a homomorphism $f : \mathbb{N} \to H$ and a $g \in H$ such that $A = A_g$;

b) if for some numbers $a,b,n$ we have $a \in A$, $b \in A$ and $an \in A$, then necessarily $bn \in A$.

**Definition 1.2.** We call a set (in any semigroup) satisfying condition b) above a quasi-ideal.

The proof of Theorem 1 was based on Davenport and Erdős' theorem and the following result of mine [4]. If $H$ is a group, then the sets $A_g$ have asymptotic densities. This result is a 'deep' one in the following sense: if $G$ is a two element cyclic group, say of the numbers $\pm 1$ and $f(p) = -1$ for all primes $p$, then it is equivalent to the prime number theorem.

Here we give an alternative proof of this theorem applying a result on convolution of measures on discrete semigroups. Let $H$ be a commutative semigroup, which we consider topologically as discrete. For one-element sets we shall use the notation $\mu(x)$ instead of $\mu(\{x\})$. For probability measures $\mu$, $\nu$ on $H$ (these are necessarily concentrated on a countable subset) we denote their convolution by $\mu \nu$. Let us recall that, by view of the discreteness, this means simply

$$\mu \nu(X) = \sum_{u,v \in H, uv \in X} \mu(u) \nu(v). \quad (1.2)$$

Then this result ([8], Theorem 1.1) is as follows.