ON CANONICAL FIBRATIONS OF ALGEBRAIC SURFACES

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Let $S$ be an algebraic surface of general type. If the canonical system $|K_S|$ of $S$ is a pencil of genus $g$, we hope to find the largest $c(g)$ such that $K_S^2 \geq c(g)p_g + \text{constant}$. We have known that $c(3) \leq 6$. In this paper, we proved that $c(3) \geq 5.25$.

Introduction

Let $S$ be a smooth projective surface of general type over $\mathbb{C}$ and $K_S$ a canonical divisor of $S$. $\phi_n$ denotes the map determined by linear system $|nK_S|$. $\phi_1$ is called canonical map of $S$, and $\phi_n (n \geq 2)$ the pluricanonical map of $S$. Much has been known about the pluricanonical map of $S$. However, the situation of canonical map is completely different, the method used in the study of pluricanonical map is not valid again. As far as I know, A. Beauville is the first man to study $\phi_1$ systematically, he studied $\phi_1$ according to $\dim \phi_1(S) = 1$ or $\dim \phi_1(S) = 2$. The proof of his theorem rests heavily on Bogomolov-Miyaoka-Yau inequality. In this paper we are only interested in the case $\dim \phi_1(S) = 1$. By the Stein factorization of $\phi_1$, we have $S \to B \to \phi_1(S)$, where $B$ is a smooth curve of genus $b$, and the general fibre $F$ of $f$ is of genus $g$. Let $p_g = \dim H^0(S, K_S), q = \dim H^1(S, \mathcal{O}_S)$ and $\chi = \chi(\mathcal{O}_S) = p_g - q + 1$. Then, according to Beauville, $f$ is a fibration and $2 \leq g \leq 5$ when $\chi(\mathcal{O}_S) \geq 21$, we call this fibration the canonical fibration of $S$. We have known that there exist surfaces with canonical fibrations of genus 2 and 3 such that $p_g$ tends to infinity. Now two questions arise: (1) Does there exist any surface of general type with canonical fibration of genus 4 and 5. G. Xiao conjectured that there is no surface of general type with canonical fibration of genus 5 when $p_g$ is large (see [6]). (2) When $S$ has canonical fibration of genus $g$, find the largest $c(g)$ such that $K_S^2 \geq c(g)p_g + \text{Constant}$. G. Xiao proved $c(2) = 4$ (which was conjectured by M. Ried), and left the estimate of $c(3)$ as an open problem (see [6]). In this paper, we obtain some partial results about the two problems as the following.
Theorem 1. If \(|K_S|\) induces a hyperelliptic fibration of genus \(g\), then

\[ K_S^2 \geq \begin{cases} \frac{2(g-1)}{3}(x + q) - 8(g - 1), & \text{if } b = 0, \\ 2(g-1)x, & \text{if } b = 1, \end{cases} \]

Corollary 1. If either \(b = 1\) or \(p_2 \geq 53 - 15q\), then \(S\) has no hyperelliptic canonical fibration of genus 5.

Theorem 2. Let \(f : S \to B\) be the canonical fibration with reduced fibres of genus 3, then

\[ K_S^2 \geq \frac{5}{4}p_2 - \frac{71}{6}. \]

Throughout the paper, we adopt the notation in [2] and always assume the smooth projective surface \(S\) is of general type and minimal over \(\mathcal{C}\).

1. The hyperelliptic case

Let \(|K_S|\) be a pencil of genus \(g\) and \(p_2 > 2g - 2\), then \(\phi_1\) has no base point, i.e. \(f : S \to B\) is a fibration of genus \(g\). By Xiao's estimate of \(b\) and \(q\) (\(b = 0, q < 2\) or \(b = q = 1\)), we can write

\[ K_S \equiv Z + f^*D_a, \quad a = p_2 + b - 1, \]

where \(Z\) is the fixed part of \(|K_S|\), \(D_a \in \text{Pic}(B)\) and \(a = \deg(D_a)\).

Let \(Z = H + V\) and \(H = \sum_{i=1}^{k} n_i C_i\), where \(V\) is the vertical part of \(Z\) (contained in some fibres of \(f\)) and \(C_i\) is the irreducible component of \(H\), which is the horizontal part of \(Z\).

We can assume \(n_1 \geq n_2 \geq \cdots \geq n_k\) and \(F\) the general fibre of \(f\). Now we consider the case that \(f\) is hyperelliptic fibration of genus \(g\), i.e. \(F\) is a hyperelliptic curve. We have an involution \(\sigma\) on \(S\) such that the following diagram is commutative.

\[
\begin{array}{ccc}
\tilde{S} & \xrightarrow{\pi} & P = \tilde{S}/\sigma \\
\varepsilon \downarrow & & q \downarrow \\
S & \xrightarrow{\sigma} & B
\end{array}
\]

where \(P\) is the ruled surface over \(B\), \(\varepsilon\) is the composition of some blowing-ups with the centers of isolated fixed points of \(\sigma\), \(\pi\) is the double cover induced by \(\sigma\).

If \(n_1 \geq g\), then \(C_1\) is a section of \(f\). Let \(P_F = C_1 \cap F\), it is easy to see that \(h^0(gP_F) = \dim R^0(F, gP_F) \geq 2\) by the Riemann-Roch theorem since \(K_F = (K_S + F)|_F \geq gP_F\). Thus \(P_F\) is the canonical ramification point of \(F\), which implies \(P_F\) is the fixed point of \(\sigma\).

Now we have proved that \(C_1\) is a component of the fixed locus of \(\sigma\) and \(C_1\) does not