FIELD-DEPENDENT HOMOLOGICAL BEHAVIOR
OF FINITE DIMENSIONAL ALGEBRAS*

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It is shown that the little finitistic dimension of a finite dimensional algebra, i.e.,
the supremum of the finite projective dimensions attained on finitely generated
modules, is not necessarily attained on a cyclic module. In general, arbitrarily
high numbers of generators are required. Moreover, it is demonstrated that this
phenomenon may depend on the base field $k$. In fact, for each integer $d \geq 3$, there
exists a quiver $\Gamma$ with a set $\rho$ of paths such that the little finitistic dimension of the
finite dimensional algebra $k\Gamma/\langle \rho \rangle$ is attained on a cyclic module precisely when
$|k| \geq d$. By contrast, the global dimension of finite dimensional monomial relation
algebras does not depend on the base field.

1. Introduction and notation
From a spurt of recent work it has become apparent that, also in non-commuta-
tive situations, the finitistic dimensions introduced by Auslander-Buchsbaum [2]
and Bass [4] are highly informative invariants which are, moreover, accessible in
many interesting contexts, including non-artinian situations such as enveloping
algebras of Lie algebras and rings of generic matrices (see, e.g., [3], [8–16]). In
particular, given a finite dimensional algebra $\Lambda$ over a field $k$, the combination of
the following three dimensions tells much of the algebra's homological tale:

\[
\begin{align*}
cyc \text{fin dim } \Lambda &= \sup \{ p \dim M \mid M \text{ is a cyclic left} \\
& \quad \Lambda\text{-module with } p \dim M < \infty \} \\
\text{fin dim } \Lambda &= \sup \{ p \dim M \mid M \text{ is a finitely generated left} \\
& \quad \Lambda\text{-module with } p \dim M < \infty \} \\
\text{Fin dim } \Lambda &= \sup \{ p \dim M \mid M \text{ is an arbitrary left} \\
& \quad \Lambda\text{-module with } p \dim M < \infty \}
\end{align*}
\]

where $p \dim M$ denotes the projective dimension of $M$. We refer to these invariants
as the cyclic, little and big finitistic dimensions of $\Lambda$, respectively.

In [13] the author showed that the big finitistic dimension may exceed the little;
however, so far no examples had been known where the little finitistic dimension is

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not attained on a cyclic module. While it may not come as a surprise that fin dim $\Lambda$, in turn, can be strictly larger than cyc fin dim $\Lambda$, it was completely unexpected that this phenomenon may depend on the cardinality of the base field. More precisely, starting with any natural number $d \geq 3$, we exhibit a quiver $\Gamma_d$ and a set $\rho_d$ of paths in $\Gamma_d$, resulting in a monomial relation algebra $\Lambda_d = k\Gamma_d/\langle \rho_d \rangle$ of finite dimension over any base field $k$, such that cyc fin dim $\Lambda_d < $ fin dim $\Lambda_d$ precisely when $|k| < d$. Going still further in this direction, we construct for each finite field $k$ and each natural number $m \geq 2$ a finite dimensional monomial relation algebra $\Lambda$ over $k$ such that fin dim $\Lambda$ is not attained on any module with fewer than $m$ generators, whereas fin dim $K \otimes_k \Lambda = cyc fin dim K \otimes_k \Lambda$ for every extension field $K$ of $k$ of sufficiently high cardinality. Amusingly, the extent to which the base field can influence the various finitistic dimensions of a monomial relation algebra is limited to adding a term $\pm 1$, although we expect this to be no longer true in non-monomial situations. These phenomena are all the more surprising as the global dimension of monomial relation algebras is not affected by the choice of field (see [11, p. 178] or Observation A below).

The inequality 'cyc fin dim $\Lambda < $ fin dim $\Lambda$' can also be realized in a field-independent fashion; namely, there exist quivers $\Gamma$ and sets $\rho$ of paths in $\Gamma$ such that, irrespective of the choice of $k$, the cyclic finitistic dimension of $k\Gamma/\langle \rho \rangle$ is strictly smaller than the little finitistic dimension. On the other hand, we know of no finite dimensional monomial relation algebra $\Lambda$ over an infinite field for which fin dim $\Lambda$ is not attained on a module with at most three generators.

To understand the bearing of the base field on the homology of a finite dimensional algebra appears as a rather pressing necessity at this point. Indeed, it is required to assess the theoretical potential of the increasingly sophisticated algorithms which are being developed for the computation of minimal projective resolutions in the case of a finite field, on the basis of, e.g., [6].

All of our examples are assembled from a small collection of building blocks which we present in a preliminary section as a 'construction kit'. In particular, we invite the reader to modify our constructions on the basis of this kit.

**Notation and Prerequisites.** Throughout, $\Lambda = k\Gamma/I$ will denote a finite dimensional monomial relation algebra over a field $k$, that is, $\Gamma$ is a quiver and $I$ an ideal of the corresponding path algebra $k\Gamma$ which is generated by a set of paths of length $> 2$ in $\Gamma$. Our convention for composing paths in $\Gamma$ is as follows: $\gamma'$ stands for '$\gamma$ after $\alpha$'.

The Jacobson radical of $\Lambda$, denoted by $J$, clearly has the following canonical vectorspace basis $P$ over $k$. For $j \geq 0$ let $P_j$ be the set of all nontrivial paths in $\Lambda$ of length $j$, i.e., of all nonzero residue classes in $\Lambda$ of paths of length $j$ in $\Gamma$; this terminology is unambiguous since $\Lambda$ is a monomial relation algebra. If $J^{L+1} = 0$, define $P = \bigcup_{1 \leq j \leq L} P_j$, that is, $P$ is the set of nontrivial paths in $\Lambda$ of length $\geq 1$. Moreover, note that $P_0$ is just the set of primitive idempotents of $\Lambda$. The primitive idempotent corresponding to a vertex $v$ of $\Gamma$ will be written as either $e_v$ or $e(\ast)$.

The first syzygy of a left $\Lambda$-module $M$, that is, the kernel of a projective cover of $M$, will be denoted by $\Omega^1(M)$, the higher syzygies by $\Omega^{k+1}(M) = \Omega^1(\Omega^k(M))$.

Finally, we recall some results from [13] which will be repeatedly applied in the sequel. Given a left $\Lambda$-module $X$, call $x \in X$ a top element of $X$ if $x \notin JX$ and $ex = x$ for some primitive idempotent $e \in P_0$; in that case $x$ will also be called a top element of type $e$. Moreover, denote by $P(X)$ the set of all those paths of length