COUNTEREXAMPLES TO THE CONJECTURE
ON MINIMAL $S^2$ IN $CP^n$
WITH CONSTANT KAELER ANGLE

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ABSTRACT. In this paper we give three families of counterexamples to the conjecture made by J.Bolton et al. in 1987: a minimal 2-sphere $S^2$ in $CP^n$ with constant Kaehler angle $\theta \neq 0, \pi, \frac{\pi}{2}$ has to be the round sphere $S^2$.

§1 INTRODUCTION

As pointed out by Chern and Wolfson[2], the Kaehler angle of an immersed real surface in a Kaehler manifold is the most fundamental invariant. In [1] J.Bolton, G.R.Jensen, M.Rigoli and L.M.Woodward shown that if $\psi: S^2 \rightarrow CP^n$ is a minimal immersion such that the induced Gaussian curvature $K$ is constant, then $\psi$ is an element of the Veronese sequence, at most up to a holomorphic isometry of $CP^n$. Therefore $K$ is constant implies the Kaehler angle $\theta$ is constant for a minimal $S^2$ in $CP^n$. Conversely, they shown that the Kaehler angle $\theta$ is constant ($\neq 0, \pi, \frac{\pi}{2}$) implies the Gaussian curvature $K$ is constant in two situations: (1) $n \leq 4$ or (2) both $n$ and $n+2$ are prime integers and $\psi$ is totally unramified. So they conjectured that the conclusion holds true in general.

Since there is only one complex structure on $S^2$, any immersion $\psi: S^2 \rightarrow CP^n$ can be conformal by a suitable diffeomorphism of $S^2$, and if $\psi$ is not linearly full one can find a totally geodesic $CP^1 \subset CP^n$ such that $\psi: S^2 \rightarrow CP^1$ is linearly full. Thus we can restate the original conjecture as follows.

Conjecture[1]. Let $\psi: S^2 \rightarrow CP^n$ be a minimal immersion with constant Kaehler angle $\theta \neq 0, \pi, \frac{\pi}{2}$, i.e., $\psi$ is neither holomorphic, anti-holomorphic nor totally real. Then the Gaussian curvature $K$ of the induced metric on $S^2$ is constant.

For each linearly full minimal immersion $\psi: S^2 \rightarrow CP^n$ there is an associated harmonic sequence $\psi_0, \psi_1, \cdots, \psi_n$ with $\psi = \psi_k$ for some $k = 0, 1, \cdots, n$, where $\psi_0$ is holomorphic, called the directrix of $\psi$(see[3,4]). Following [4] we call $\psi$ a minimal immersion with position $k$. Our main results is

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Theorem A. Let $\psi: S^2 \to CP^n$ be a linearly full minimal immersion with position 2. Suppose the Kaehler angle $\theta$ is constant but the Gaussian curvature is not. If $\theta \neq 0, \pi, \frac{\pi}{2}$ and the directrix $\psi_0$ of $\psi$ is unramified, then $n \leq 10$ and $tg^2(\theta/2) = 3/4$.

Theorem B. There are at least three families of totally unramified minimal immersions $\psi: S^2 \to CP^{10}$ such that $\psi$ is neither holomorphic, anti-holomorphic nor totally real, with constant Kaehler angle and nonconstant Gaussian curvature. Moreover, $\psi$ is homotopic to the Veronese minimal immersion.

In order to prove these results we need some algebraic lemmas appeared in §2 and give several formulas by moving frames to compute the Kaehler angle and the Gaussian curvature in §3 and §4. Then in §5 we give the proof of Theorem A and B.

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§2 ALGEBRAIC LEMMA

The canonical symmetric inner product $\langle , \rangle$ in $C^n$ induces naturally the canonical symmetric inner product in $\wedge^j C^n = C^n \wedge \cdots \wedge C^n \cong C^N$ ($N = \binom{n}{j}$) by

$$\langle u_1 \wedge \cdots \wedge u_j, v_1 \wedge \cdots \wedge v_j \rangle = \det(\langle u_j, v_k \rangle)$$

for $u_j, v_k \in C^n, j, k = 1, \ldots, l$. According to the rule of determinant expansion we have

$$\langle u_1 \wedge \cdots \wedge u_j, v_1 \wedge \cdots \wedge v_j \rangle = \sum_{j=1}^{l} (-1)^{i-j} \langle u_1 \wedge \cdots \wedge \hat{u_j} \cdots \wedge u_i, v_1 \wedge \cdots \wedge v_{i-1} \rangle \langle u_j, v_i \rangle$$

(2.1)

where $u_1 \wedge \cdots \hat{u_j} \cdots \wedge u_l = u_1 \wedge \cdots \wedge u_{j-1} \wedge u_{j+1} \wedge \cdots \wedge u_l$.

Lemma 1. If $u_1, \ldots, u_{l+2}, v_1, \ldots, v_{l+2} \in C^n$, then

$$\langle u_1 \wedge \cdots \wedge u_l \wedge u_{l+1}, v_1 \wedge \cdots \wedge v_l \wedge v_{l+1} \rangle \langle u_1 \wedge \cdots \wedge u_{l+2}, v_1 \wedge \cdots \wedge v_{l+2} \rangle$$

$$= \langle u_1 \wedge \cdots \wedge u_l, v_1 \wedge \cdots \wedge v_l \rangle \langle u_1 \wedge \cdots \wedge u_{l+1} \wedge u_{l+2}, v_1 \wedge \cdots \wedge v_{l+1} \wedge v_{l+2} \rangle.$$

Proof. Set

$$u = \sum_{j=1}^{l+2} (-1)^{i+j} \langle u_1 \wedge \cdots \hat{u_j} \cdots \wedge u_{l+2}, v_1 \wedge \cdots \wedge v_{l+1} \rangle u_j$$

Then by (2.1) $\langle u, v_j \rangle = 0$ for $j = 1, \ldots, l+1$ and

$$\langle u, v_{l+2} \rangle = \langle u_1 \wedge \cdots \wedge u_{l+2}, v_1 \wedge \cdots \wedge v_{l+2} \rangle$$

Again by (2.1) and the bilinearity we get

$$RHS = \langle u_1 \wedge \cdots \wedge u_l \wedge u, v_1 \wedge \cdots \wedge v_l \wedge v_{l+2} \rangle = LHS.$$