Some loci in Teichmüller space for genus seven defined by vanishing thetanulls*

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Let $\theta[e](u)$ be a theta function for a Riemann surface $W$ of genus seven. Suppose $\theta[e](u)$ vanishes at $u = 0$ for 4 half-integer theta characteristics $[\varepsilon_i]$, $i = 1, 2, 3, 4$ to orders 2, 2, 2, and 3 respectively and $(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4) = (0)$. Then $W$ is hyperelliptic, elliptic-hyperelliptic or $W$ lies in the closure of a locus in Teichmüller space of Riemann surfaces which admit plane models where the four half-canonical linear series corresponding to the theta-vanishings are clearly evident.

1. Introduction

Let $W_p$ be a compact Riemann surface of genus $p$. Let $T_p$, $H_p$, $(E - H)_p$ stand for, respectively, Teichmüller space for genus $p$, the hyperelliptic locus in $T_p$, and the elliptic-hyperelliptic locus in $T_p$. Our primary interest is the case $p = 7$. Then $H_7$ has pure codimension 5 in $T_7$, and $(E - H)_7$ has pure codimension 6. In this paper we shall give local defining equations for $H_7$, $(E - H)_7$, and a third locus, $N_7$, in terms of the vanishing properties of the theta function.

The vanishing properties are as follows. Let a canonical homology basis be chosen for $W_7$, and so corresponding theta functions with half-integer theta characteristics, $\theta[e](u)$, are defined. Let $[\varepsilon_i]$, $i = 1, 2, 3, 4$, be theta characteristics so that (i) $\theta[e_i](u)$ vanishes at $u = 0$ to orders 2, 2, 2, and 3 for $i = 1, 2, 3, 4$, and (ii) $(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4) = (0)$.

By Riemann's vanishing theorem ([12, p. 459], and [13]), these vanishing properties are equivalent to the existence on $W_7$ of 4 complete half-canonical linear series, $g^1_6$, $h^1_\delta$, $k^1_\delta$, and $\mathcal{L}_c^2$ whose sum is bicanonical. Such a set of 4 half-canonical linear series will be called a quartet and $\mathcal{L}_c^2$ will be called the leader. We shall state our theorem in terms of the existence of a quartet.

By the known vanishing properties of theta functions for hyperelliptic and elliptic-hyperelliptic Riemann surfaces of genus 7 ([12, p. 459], [10, Ch. VII], and [3, p. 51]) many

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quartets exist on such surfaces. The third locus, \( \mathcal{N}_7 \), will be the closure in \( T_7 \) of Riemann surfaces which have a distinctive plane model which will now be described.

Let \( A_1, A_2, A_3 \) and \( A_4 \) be the 4 points in \( \mathbb{P}^2(C) \), \( (\pm 1, \pm 1, 1) \). Let \( P, Q \) and \( R \) be the points \( (1, 0, 0), (0, 1, 0), \) and \( (0, 0, 1) \), the 3 diagonal points of the quadrangle \( A_1A_2A_3A_4 \).

Let \( C_9 \) be a plane curve of degree 9 with singularities only at these 7 points, and each singularity is an ordinary singularity of multiplicity 3, or any other singularity of multiplicity 3 which contributes 3 to the \( \delta \)-invariant of \( C_9 \). \( C_9 \) has genus 7 and the dimension of such curves, \( C_9 \), in \( T_7 \) is 12. The quartet is as follows. The leader, \( \ell_4 \), is cut out by cubics passing through all 7 singular points. The other 3 linear series are cut out by lines passing through the 3 diagonal points \( P, Q, \) and \( R \). The closure in \( T_7 \) of Riemann surfaces admitting such models will be denoted \( \mathcal{N}_7 \). Models of the above type will be called \textit{general} in contradistinction to \textit{non-general} models now to be described.

Consider a general model as described in the preceding paragraph. Perform an elementary quadratic transformation with fundamental points \( A_2, A_3, \) and \( A_4 \). The transformed curve, \( C'_9 \), is again of degree 9 with an ordinary singularity of multiplicity 3 at \( A'_1 \) (the transform of \( A_1 \)) and with (what we shall call) a \( (3, 6) \)-point at \( P', Q', \) and \( R' \) (the transforms of \( P, Q, \) and \( R \)).

**Definition.** A \( (3, 6) \)-point is a singularity of multiplicity 3 where all branches have a common tangent (the \( (3, 6) \)-tangent) which intersects the curve at least 6 times at the singularity. (A \( (3, 6) \)-point will contribute at least 6 to the \( \delta \)-invariant.)

On \( C'_9 \) all 3 of the \( (3 - 6) \)-tangents will pass through \( A'_1 \). The points \( P', Q', \) and \( R' \) are not collinear. \( \ell'_4 \) is cut out by cubics through all 4 singularities and tangent at \( P', Q', \) and \( R' \) to the \( (3-6) \)-tangents. \( g'_6 \) will be cut out by conics through \( P', Q', \) and \( R' \) and tangent to the \( (3-6) \)-tangent at \( P' \). Similarly for \( h'_6 \) and \( k'_6 \).

But for a plane curve like \( C'_9 \), there is no reason why the 3 points \( P', Q', \) and \( R' \) cannot be collinear. If they are, the corresponding \( W_7 \) will be said to admit a \textit{non-general} model. In this case \( \ell'_6 \) is cut out as before, but the other linear series are now cut out by lines passing through \( P', Q', \) and \( R' \).

We shall also see that in \( \mathcal{N}_7 \) there are \( W_7 \)'s which are 3-sheeted coverings of tori. Any 3-sheeted covering of a torus will be called \textit{elliptic-trigonal} and the locus of such surfaces in \( T_7 \) will be denoted \( (E-T)_7 \). By the Riemann-Hurwitz formula, each component of \( (E-T)_7 \) has dimension 12.

We now state the theorem.

**Theorem.** Suppose \( W_7 \), a Riemann surface of genus 7, admits 4 complete half-canonical linear series \( g'_6, h'_6, k'_6, \) and \( \ell'_6 \) whose sum is bicanonical. Then one of the following possibilities holds:

(i) \( W_7 \) is hyperelliptic
(ii) \( W_7 \) is elliptic-hyperelliptic
(iii) \( W_7 \) is in \( \mathcal{N}_7 \).