Bifurcation near solutions of variational problems with degenerate second variation

Xianqing Li-Jost
Mathematisches Institut, Ruhruniversität, 44780 Bochum

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We investigate the bifurcation behavior of solutions of variational problems, in particular minimal surfaces, for which the first eigenvalue of the second variation vanishes while the third variation is nonzero.

1. Introduction

In this paper we treat a general variational problem with a critical point with smallest eigenvalue of the second variation zero and whose third variation in an eigenfunction direction does not vanish. There are then bifurcations, which means that there are two branches of critical points of the variational integral if we vary the boundary a little bit, one branch minimizing and the other one not. This was first investigated by L. Lichtenstein [LL] in a 2-dimensional Euclidean situation. He mainly used potential theory to solve this problem. We obtain bifurcations near the critical point by using the theorems about inverse and implicit functions and we can treat this kind of variational problem on manifolds.

For motivation, we first state the following corollary of our main result:

Theorem. Let $N$ be an $(n+1)$-dimensional orientable complete Riemannian manifold with metric $\gamma = (\gamma_{ij})$. $M$ denotes an $n$-dimensional compact orientable manifold with $\partial M \neq \emptyset$, $\partial M \in C^2$.

Assume there is a minimal immersion $i : M \rightarrow N$ (for simplicity of notation, we identify $M$ with $i(M)$ in this paper). with smallest eigenvalue zero, while the third variation of the volume integral at $i$ in one eigenfunction direction $\phi_0 n$ is not zero (for the definitions see page 13 and formul (27)).
Then

1) \( M \) is not a local minimum of the volume integral.

2) There are two families of minimal immersions \( M^+_\epsilon \) and \( M^-\epsilon \) nearby \( M \), and all the immersions of one family are locally area minimizing and all those of the other family are unstable.

In order to put this result into perspective, the following remarks might be appropriate:

Usually, one produces minimal surfaces by minimizing or minimaxing, i.e. one seeks stable or unstable critical points of the area functional, see e.g. the discussion and references in [Jo] and [St]. In "nondegenerate" or "generic" situations, however, stable minimal surfaces have positive and unstable ones have negative first eigenvalue. Cases where the first eigenvalue vanishes are typically much more difficult to analyze than the nondegenerate ones because in principle after a small perturbation minimal surfaces could thus exhibit qualitatively quite different behavior. In fact, many of the difficulties in the theory of closed geodesics stem from the necessity to include such degenerate situations, see [Kl]. Our result shows that under the additional assumption that the third variation is nonzero, one may still control the qualitative behavior. A good example where our assumptions are satisfied is the case of a catenoid, a minimal surface in \( \mathbb{R}^3 \). Take two coaxial circles of the same radius \( R \). If their distance \( h \) is small compared to \( R \), they bound two connected minimal surfaces, a stable and an unstable catenoid. If the distance reaches a critical value \( h(R) \) (computed in [Ni]), then they bound precisely one catenoid with zero first eigenvalue and nonvanishing third variation. If \( h > h(R) \), the two circles do not bound any connected minimal surface. This example is discussed in detail in the light of our theorem in [W].

There are other examples of minimal surfaces in \( \mathbb{R}^3 \) with zero first eigenvalue where our assumption of nonvanishing third variation is not satisfied, e.g. Enneper's minimal surface. This example was analyzed in [BT]. It is, however, possible to develop a general theory of minimal surfaces with zero first eigenvalue and some nonvanishing higher order variation. In fact, in [JLJP] a generalization of our theory is presented showing that the bifurcation of such minimal surfaces can be described in terms of "elementary catastrophes". Let us also quote [B] where a different approach to unstable minimal surfaces is pursued, namely an analysis in terms of the Weierstrass representation. By its very nature, such an approach is restricted to Euclidean space as ambient space, whereas our methods work in arbitrary Riemannian manifolds.

2. Notations

We'll investigate the general variational integral

\[
I(u) = \int_M F(x, u, \nabla u) dM
\]  

(1)

for \( u \in H^{1,2}(M) \) where