Minimal forms with respect to function fields of conics

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Received April 26, 1994

Summary. Let $F$ be a field of characteristic $\neq 2$, and let $\rho$ be an anisotropic conic over $F$. Anisotropic quadratic forms $\varphi$ over $F$ which become isotropic over the function field $F(\rho)$, but which do not contain proper subforms becoming isotropic, are called $F(\rho)$-minimal forms. It is investigated how upper bounds for the dimension of $F(\rho)$-minimal forms depend on certain properties and invariants of the field $F$. The existence of fields $F$ and conics $\rho$ such that $F$ contains $F(\rho)$-minimal forms of arbitrarily large (odd) dimension is proved.

0. Introduction

Let $F$ be a field of characteristic $\neq 2$ and $\rho = (1, a, b)$ an anisotropic conic over $F$. A well-known consequence of the Cassels-Pfister subform theorem, cf. [13, Ch. 4, Theorem 5.4], states that an anisotropic form $\varphi$ over $F$ becomes hyperbolic over the function field $F(\rho)$ of the conic $\rho$, if and only if

$$\varphi \cong_F \alpha \otimes \langle (a, b) \rangle,$$

for some form $\alpha$ over $F$. (Here $\langle (a, b) \rangle = (1, a, b, ab)$.) The question which anisotropic forms over $F$ can become isotropic over the function field $F(\rho)$ of a conic was raised by Lam in the seventies and regained interest after the work of Merkurjev on the $u$-invariant, cf. [10, page 27]. Now if an anisotropic form over $F$ becomes isotropic over $F(\rho)$ then it may possibly contain a subform of smaller dimension which already becomes isotropic over $F(\rho)$. This observation led the first author to introduce the following notion in his thesis [5]:

\footnote{During the work on this article, the first author was a postdoc at the Institute for Experimental Mathematics, University of Essen, Germany, supported by a grant from the Deutsche Forschungsgemeinschaft}
A quadratic form $\varphi$ over $F$ is an $F(\rho)$-minimal form if the following conditions are fulfilled:

i. $\varphi$ is anisotropic

ii. $\varphi \otimes F(\rho)$ is isotropic

iii. $\psi \otimes F(\rho)$ is anisotropic for any subform $\psi \subset \varphi$ with $\dim \psi < \dim \varphi$.

(In [5, Def. 2.6.1] the notion is defined for an arbitrary field extension $K/F$.) It is clear that "characterising" all $F(\rho)$-minimal forms would answer the question of Lam. Originally Lam asked whether or not it is true that the only $F(\rho)$-minimal forms $\varphi$ are the forms similar to $\rho$. The answer is yes for forms $\varphi$ of dimension $\leq 4$ (proved by Shapiro in his thesis) and for 6-dimensional Albert forms (proved by Merkurjev, cf. [10, Sec. 6]). Soon after Lam raised the question Wadsworth produced examples of 5-dimensional $F(\rho)$-minimal forms for suitably chosen $F, \rho$, thus providing a negative answer to Lam's original question. It turns out that $F(\rho)$-minimal forms are necessarily of odd dimension (cf. Remark 1.7). The first 7-dimensional example of an $F(\rho)$-minimal form can be found in [5, Sec. 3.5].

In this paper we first investigate how upper bounds for dimensions of possible $F(\rho)$-minimal forms depend on the invariants of the field $F$ (sections 2 and 3). In section 2 some general results are given and in section 3 we consider 2-henselian ground fields $F$.

However, our main result (section 4) is a proof of the existence of fields $F$ and conics $\rho$ such that $F$ contains $F(\rho)$-minimal forms of arbitrary large (odd) dimension. The construction uses Laurent series and is based on the results obtained in section 3. This indicates that finding a handy characterisation of all $F(\rho)$-minimal forms for general fields $F$ will be a very difficult venture so that a full and satisfactory answer to Lam's question seems still out of reach.

The proofs in sections 3 and 4 depend heavily on the notions splitting sequence and splitting index as introduced in [11]. In [6] the relation between splitting sequences and $F(\rho)$-minimal forms has been worked out. It is also shown there that results on $F(\rho)$-minimal forms can be understood better from this relationship. In section 1 we settle notation, terminology and recall the properties and theorems used further in the paper.

1. Notations and facts

All the fields that occur are of characteristic not 2. We adopt the notations and terminology used in the standard references [9] and [13].

With $\varphi \perp \psi$, $\varphi \cong \psi$, $\varphi \subset \psi$, and $\varphi = \psi$ we denote respectively orthogonal sum of forms, isometry of forms, $\varphi$ is isometric to a subform of $\psi$, and the class of $\varphi$ equals the class of $\psi$ in the Witt ring $WF$ of the field $F$. The hyperbolic plane $(1, -1)$ is denoted by $\mathbb{H}$ and $i_W(\varphi)$ stands for the Witt index of $\varphi$. Thus, if we write $\varphi_{an}$ for the anisotropic part of $\varphi$ and if $i_W(\varphi) = n$ then $\varphi \cong \varphi_{an} \perp n\mathbb{H}$. We call $\varphi$ similar to $\psi$ if $\varphi \cong a\psi$ for some $a \in F^* (= F \setminus \{0\})$. With $d(\varphi) \in F^*/F^2$ we denote the discriminant of $\varphi$ and with $c(\varphi) \in Br_2(F)$ its Clifford invariant.

The symbol $\langle a_1, \ldots, a_n \rangle$ denotes the $n$-fold Pfister form $(1, a_1) \otimes \cdots \otimes (1, a_n)$, and $P_n F$ is the set of all $n$-fold Pfister forms. Such forms $\tau \in P_n F$ are multiplicative, i.e., they are isotropic if and only if they are hyperbolic and the sets $D(\tau) = D_F(\tau) = \{a \in F^* | (a) \subset \tau\}$.