ARRANGEMENTS WITH ONE BOUNDED COMPONENT
AND $p$-ADIC INTEGRALS

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In this article we study arrangements $A$, such that $\mathbb{R}^n \setminus A$ has exactly one bounded component. We obtain a result about their structure which gives us a method to construct all combinatorially different such arrangements in a given dimension. (A complete list for dimensions 1, 2, 3 and 4 is included).

Furthermore we associate a $p$-adic integral to each such arrangement and prove that this integral can be written as a product of $p$-adic beta functions. This is analogous to results of Varchenko and Loeser for integrals over $\mathbb{R}$ and character sums over finite fields respectively.

Introduction

We study arrangements $A$ of hyperplanes in $\mathbb{C}^n$ with real coefficients such that $\chi(\mathbb{C}^n \setminus A) = \pm 1$, where $\chi$ denotes the Euler-Poincare characteristic. If the normal vectors to the hyperplanes of $A$ span $\mathbb{C}^n$, the assumption $\chi(\mathbb{C}^n \setminus A) = \pm 1$ is known to be equivalent with the number of bounded components of $\mathbb{R}^n \setminus A$ being one. If the latter condition is satisfied, $A$ will be called a real 1-arrangement.

To each real 1-arrangement, we associate a $p$-adic integral. Theorem A states that such integrals can be written as a product of beta functions. Analogous results have been proved in [V] for real integrals and in [L] for character sums over finite fields. To prove theorem A, we need to know the structure of real 1-arrangements. The main result concerning this structure is stated in theorem B. We will see how this theorem implies that the number of hyperplanes $\#\mathcal{H}(A)$ of $A$ is bounded by $n + 1 \leq \#\mathcal{H}(A) \leq 2n$. Moreover, it gives a method to construct all real 1-arrangements of a given dimension. For lower dimensions, the construction is explicitly done. We also give a few examples of theorem A. Concerning the structure of the article, we have separated the main results, their proofs and the examples into 3 paragraphs.

Finally we'd like to thank Prof. J. Denef and Prof. F. Loeser for suggesting the problem and J. Denef for teaching us the necessary background material. We remark that the integrals we study in this article are related to Igusa's local zeta function [I] [D].
1. Announcement of the main results

1.1. Notations and definitions

We define a $n$-dimensional arrangement $\mathcal{A}$ to be a finite union of hyperplanes in $\mathbb{C}^n$. $\mathcal{A}$ will be called real if all the coefficients of the equations of the hyperplanes can be taken real and generating if the normal vectors to the hyperplanes span $\mathbb{C}^n$.

Let $\mathcal{A}$ be a $n$-dimensional real arrangement. By $BC(\mathbb{R}^n \setminus \mathcal{A})$ we will indicate the set of all bounded components of $\mathbb{R}^n \setminus \mathcal{A}$ and we will call $\mathcal{A}$ a real $k$-arrangement if $\#BC(\mathbb{R}^n \setminus \mathcal{A}) = k$. For an arbitrary arrangement $\mathcal{A}$, $\mathcal{H}(\mathcal{A})$ will denote the set of all hyperplanes of $\mathcal{A}$. To each hyperplane $H_r \in \mathcal{H}(\mathcal{A})$ we associate 2 arrangements, i.e. $\mathcal{A}'_{H_r} = \bigcup_{H \notin \mathcal{H}(\mathcal{A}) \setminus \{H_r\}} H$ the deleted arrangement of $\mathcal{A}$ induced by $H_r$ and $\mathcal{A}''_{H_r} = \bigcup_{H \notin \mathcal{H}(\mathcal{A}) \setminus \{H_r\}} (H \cap H_r)$ the restricted arrangement of $\mathcal{A}$ to $H_r$. Let $\mathcal{A}$ be a real arrangement and $C$ a component of $\mathbb{R}^n \setminus \mathcal{A}$ defined by the inequalities $f_1 \leq 0, ..., f_m \leq 0$ where for each $i$, $\Theta_i \in \{<, \geq\}$ and $f_i = 0$ is an equation of a hyperplane $H_i$ of $\mathcal{A}$. $H_i$ will be a boundary of $C$ if the component $C$ 'changes when we omit the inequality $\Theta_i$ from the inequalities of $C$. By a boundary of a $n$-dimensional real $1$-arrangement $\mathcal{A}$, we mean a boundary of the unique bounded component of $\mathbb{R}^n \setminus \mathcal{A}$. The set of all boundaries of $\mathcal{A}$, will be denoted by $\beta(\mathcal{A})$. Finally we define two arrangements $\mathcal{A}$ and $\mathcal{B}$ to be combinatorially equivalent if there exists an inclusion preserving bijection between $L(\mathcal{A})$ and $L(\mathcal{B})$ where $L(\mathcal{A})$ denotes the set of all non-empty intersections of hyperplanes in $\mathcal{A}$.

1.2. The main results

**Theorem A.** Let $\mathcal{A} = \bigcup_{i=1}^m H_i$ be a $n$-dimensional real arrangement such that

(i) each of the $H_i$ has rational coefficients

(ii) $\chi(\mathbb{C}^n \setminus \mathcal{A}) = \pm 1$

(iii) $\mathcal{A}$ is generating.

Let $H_i$ be given by the equation $a_i(x_1, ..., x_n) = 0$. Then

$$\int_{Q_p^n} \prod_{i=1}^m |a_i(x_1...x_n)|_p^{s_i-1} \prod_{i=1}^n |dx_i|_p$$

converges on a non-empty open set $D$ of $\mathbb{R}^n$ and can be written on $D$ as

$$p^{L(s)} \prod_{i=1}^n B(L_{i1}(s), L_{i2}(s))$$

where $B$ is the $p$-adic beta function defined as in [S-T] namely $B(r, t) = \int_{Q_p} |x|_p^{r-1} (1-x|_p^t) dx|_p$; $s$ denotes $(s_1, ..., s_m)$; $L(s), L_{i1}(s), L_{i2}(s) \in \mathbb{Z}[s]$ and $\deg L(s) \leq 1, \deg L_{i1}(s) = \deg L_{i2}(s) = 1$.

The proof of theorem A depends on the following result about the structure of real 1-arrangements.