The Newton polygon of a product of power series

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Summary. Let $F$ be a field with a non-trivial valuation $v : F \to \mathbb{R} \cup \{+\infty\}$. To any power series in one variable over $F$ one can associate a Newton polygon with respect to this valuation. Let $N_1$ and $N_2$ be polygons which arise as Newton polygons of power series over $F$. We determine the set of polygons $N$ with the property that there exist power series $f_i$ with respective Newton polygon $N_i$, $i = 1, 2$, such that the product $f_1f_2$ has Newton polygon $N$.

1. Introduction

Let $F$ be a field, and let $v : F \to \mathbb{R} \cup \{+\infty\}$ be a non-trivial valuation on $F$, i.e. a map with the following properties:

1. $v(a) = +\infty \iff a = 0$
2. $v(ab) = v(a) + v(b)$ for all $a, b \in F$
3. $v(a + b) \geq \min\{v(a), v(b)\}$ for all $a, b \in F$

We have the usual convention $r + \infty = +\infty$ for all $r \in \mathbb{R} \cup \{+\infty\}$. Since $v$ is supposed to be non-trivial, we may assume (if necessary after scaling $v$ suitably) that $\mathbb{Z} \subseteq v(F)$. However, we do not assume $v$ to be discrete, nor do we require that $F$ be complete with respect to $v$. We remark that one can easily show that we have equality in 3 in case $v(a) \neq v(b)$. This fact will be used frequently in the proofs of our results.

Let $f(T) = \sum_{i=\infty}^{\infty} a_i T^i \in F((T))$ be a non-zero Laurent series. For each $i$ with $a_i \neq 0$ consider the vertical half-line $L_i = \{(i, y) \mid y \geq v(a_i)\}$ in the plane $\mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\}$. If $a_i = 0$ let $L_i$ be the empty set. Now let $L$ be the convex hull of $\bigcup_{i=\infty}^{\infty} L_i$. The boundary of $L$ is a polygon with countably many

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segments (possibly of infinite length). It is called the Newton polygon of \( f \), we write \( NP(f) \) for short.

Multiplying \( f \) by \( az^m, a \in F \setminus \{0\} = \hat{F} \), shifts the whole polygon horizontally (i.e. parallel to the x-axis) by \( m \) units and vertically (i.e. parallel to the y-axis) by \( v(a) \) units. The shape of the Newton polygon does not change, so that for all our purposes we will always assume that \( f \in 1 + TF[[T]] \).

Let \( \mathbb{R} = \mathbb{R} \cup \{ \pm \infty \} \) and \( \mathbb{N} = \{0, 1, 2, \ldots, \infty \} \). For \( n \in \mathbb{N} \) we define \( n + \infty = \infty \), and we extend the usual ordering on \( \mathbb{R} \) to \( \mathbb{R} \) by putting \( -\infty < r < +\infty \) for all \( r \in \mathbb{R} \). To each \( f \in 1 + TF[[T]] \) we associate a map \( f^*: \mathbb{R} \to \mathbb{N} \) as follows. If \( S \) is a segment of \( NP(f) \) which is not vertical (i.e. not parallel to the y-axis), and if \( s \) is the slope of \( S \) and \( l \in \mathbb{N} \) is the length of the projection of \( S \) onto the x-axis, then we put \( f^*(s) = l \). For all \( t \in \mathbb{R} \) which do not appear as slope of a segment we put \( f^*(t) = 0 \). Furthermore, we put \( f^*(-\infty) = 0 \) unless \( NP(f) \) consists only of the y-axis in which case we put \( f^*(-\infty) = \infty \) (note that then \( f^*(t) = 0 \) for all \( t > -\infty \)). Finally, \( f^*(+\infty) = 0 \) unless \( f \) is a polynomial in which case we put \( f^*(+\infty) = \infty \).

We furthermore define \( S(f) \) resp. \( I(f) \) to be the supremum resp. infimum of all the slopes of the segments which appear in \( NP(f) \). More precisely,

\[
S(f) = \sup\{ s \in \mathbb{R} | f^*(s) \neq 0 \}, \quad I(f) = \inf\{ s \in \mathbb{R} | f^*(s) \neq 0 \}.
\]

**Examples.** Let \( F = \mathbb{Q}_p \) where \( p \) is a prime and \( v(p) = 1 \), and let \( \sigma(i) = \frac{i(i+1)}{2} \) for \( i \geq 0 \).

(i) One easily checks that for \( f(T) = \sum_{i=0}^{\infty} p^{\sigma(i)} T^i \) we get \( f^*(s) = 1 \) for \( s = 1, 2, 3, \ldots \) and \( f^*(s) = 0 \) otherwise. We have \( I(f) = 1 \) and \( S(f) = +\infty \).

(ii) The Newton polygon of \( f(T) = \sum_{i=0}^{\infty} p^{-\sigma(i)} T^i \) is "degenerate", i.e. it consists only of the y-axis. Hence, we have \( f^*(-\infty) = \infty \) and \( f^*(s) = 0 \) for \( s > -\infty \). We have \( I(f) = S(f) = -\infty \).

(iii) For \( f(T) = \sum_{i=0}^{\infty} p^{-\frac{1}{n}} T^{\sigma(i)} \) we obtain \( f^*(-\frac{1}{n}) = n \) for \( n = 1, 2, 3, \ldots \), and \( f^*(s) = 0 \) otherwise. We have \( I(f) = -1 \) and \( S(f) = 0 \).

**Remark 1.1.** (i) \( NP(f) \) has finitely many segments if and only if there exists \( s \in \mathbb{R} \) with \( f^*(s) = \infty \). In this case, \( f^*(t) \neq 0 \) for only finitely many \( t < s \), and \( f^*(t) = 0 \) for all \( t > s \), and we obviously have \( S(f) = s \).

(ii) For all \( t < S(f) \) we obviously have \( f^*(t) < \infty \).

The analytic meaning of \( S(f) \) is given by the fact that \(-S(f)\) is the radius of convergence of \( f \), i.e., \( f \) converges (resp. diverges) at \( T = a \) if \( v(a) > -S(f) \) (resp. \( v(a) < -S(f) \)). For this and additional information about the applications of Newton polygons of power series in \( p \)-adic analysis, we refer the reader to \([K, \text{Ch.4, \S 4}]\). For facts about Newton polygons of polynomials and their applications in number theory we refer to \([A, \text{\S 5}]\), \([C, \text{Ch.6, \S 3}]\), \([N, \text{Ch.2, \S 6}]\).

Now it is a well-known fact that if \( f, g \in F[T] \) are non-zero polynomials then the segments of the Newton polygon of the product \( fg \) are obtained by taking all segments of the respective Newton polygons of \( f \) and \( g \), and if in this new set there are two segments of the same slope \( s \) and with length \( l_1 \) and \( l_2 \) then these two will be replaced by one segment of slope \( s \) and length \( l_1 + l_2 \). This behaviour can be expressed in terms of \( f^* \) and \( g^* \) simply by \( (fg)^* = f^* + g^* \).