A MEAN VALUE THEOREM ON DIFFERENCES OF TWO $k$-th POWERS OF NUMBERS IN RESIDUE CLASSES

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1. Introduction.

For positive integers $k > 2$ and $n$, let $t_k(n)$ denote the numbers of ways to write $n = m^k - |r|^k$ with $(m, r) \in \mathbb{N} \times \mathbb{Z}$. To study the "average order" of this function, one considers the Dirichlet's summatory function

$$T_k(x) = \sum_{1 \leq n \leq x} t_k(n),$$

where $x$ is a large real variable. Of course, the evaluation of $T_k(x)$ is a problem of lattice point theory in the classical sense of Landau. It has been proved by Krätzel [2] that, for $k > 3$ and some small $\varepsilon > 0$,

$$T_k(x) \ll x^{1 - \varepsilon} + C_2 x^{k - \varepsilon} + C_3 \sum_{n=1}^{\infty} n^{-1-\varepsilon} \sin(2\pi n x^{k-1} + \frac{\pi}{2k}) + O(x^{\frac{1}{k}-\varepsilon}),$$

with effective constants $c_i$ depending only on $k$.

Consequently, for $k \geq 3$,

$$T_k(x) = c_1 x^{\frac{k}{2}} + c_2 x^{\frac{k-1}{2}} + \frac{O}{\Omega}(x^{\frac{1}{k}-\frac{1}{2}}).$$

Applying the modern "Discrete Hardy-Littlewood method" in the shape due to Huxley [1], Müller and Nowak [4] improved the error term to $O(x^{\frac{1}{k}+\varepsilon} (\log x)^{\frac{1}{k}})$.

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2. Subjects and results of this paper.

It is an usual idea in the theory of arithmetic functions to impose some kind of congruence conditions to some of the parameters involved. For instance, think of the prime number theorem of arithmetic progressions. See also the author’s recent paper [3], where the representations of integers as a sum of two $k$-th powers of numbers in residue classes are dealt with.

In this article we are thus going to consider representations $n = u^k - v^k$ ($u, v \in \mathbb{N}_0$) with $u$ and $v$ lying in prescribed residue classes. For integers $l_1, l_2$ and a positive integer $m$, and $k > 2$ and $n$ as earlier, define

$$t_k(l_1, l_2, m; n) = \# \{(u, v) \in \mathbb{N}_0^2 | u^k - v^k = n, u \equiv l_1 (\text{mod } m), v \equiv l_2 (\text{mod } m)\}$$

(clearly, $t_k(l_1, l_2, m; n)$ is finite for every $n \geq 1$), and again for a large real variable $x$,

$$T_k(l_1, l_2, m; x) = \sum_{1 < n < x} t_k(l_1, l_2, m; n).$$

From a pure geometric point of view, one may define

$$T_k(l_1, l_2, m; x) = \# \{(u, v) \in \mathbb{N}_0^2 | 0 < u^k - v^k \leq x, u \equiv l_1 (m), v \equiv l_2 (m)\}$$

and allow $k$ to be real.

**Theorem**: (i) For $0 \leq l_1, l_2 < m$ and $x \rightarrow \infty$,

$$T_k(l_1, l_2, m; x) = C_1(k)\left(\frac{x}{m^k}\right)^{\frac{1}{k}} + C_2(l_1, l_2, m, k)\left(\frac{x}{m^k}\right)^{\frac{1}{k^2}} + C_3(l_2, m)\left(\frac{x}{m^k}\right)^{\frac{1}{k^4}} + C_4(k)\left(\frac{x}{m^k}\right)^{\frac{1}{k^5}} F_k(l_1, m; x) + O\left((\frac{x}{m^k})^{\frac{1}{k^6}} (\log(\frac{x}{m^k}))^{\frac{11}{12}}\right),$$

where

$$F_k(l_1, m; x) = \sum_{n=1}^{\infty} n^{-\frac{1}{k}} \sin\left(\frac{2\pi n}{m} (x^{\frac{1}{k}} - l_1) + \frac{\pi}{2k}\right),$$

$$C_4(k) = \frac{k^{\frac{1}{k^2} - 1 - \frac{1}{k}}}{2^{\frac{1}{k} - 1} \pi^{1 + \frac{1}{k}}}, \quad C_3(l_2, m) = \left(\frac{1}{2} - \frac{l_2}{m}\right), \quad C_1(k) = \frac{1}{4k \cos \frac{\pi}{k}} \Gamma\left(\frac{1}{k}\right),$$

and

$$C_2(l_1, l_2, m, k) = k^{-\frac{1}{k^2}} (\zeta\left(\frac{1}{k - 1}, \frac{m + l_1 - l_2}{m}\right) + \frac{m}{m + l_1 - l_2})^{\frac{1}{k^2}} (-1 - W(l_1 \leq l_2)) - \psi\left(\frac{l_2 - l_1}{m}\right),$$

with $\zeta(s, q) = \sum_{n=0}^{\infty} (n + q)^{-s}$ (Hurwitz Zeta Function), $\psi(z) = z - \lfloor z \rfloor - \frac{1}{z}$, and $W$ the truth-function, i.e. $W(P) = 1$ if $P$ is true and $W(P) = 0$ if $P$ is false.

The $O$-constant depends only on $k$. 

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