GLOBAL REGULARITY OF THE \( \bar{\partial} \)-NEUMANN PROBLEM
ON AN ANNULUS BETWEEN TWO PSEUDOCONVEX
MANIFOLDS WHICH SATISFY PROPERTY (P)

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Let \( X \) be a complex manifold of dimension \( n \geq 3 \). Let \( \Omega_1, \Omega_2 \) be two open pseudoconvex submanifolds with smooth boundary such that \( \Omega_1 \Subset \Omega_2 \Subset X \). Let \( \Omega = \Omega_2 \setminus \overline{\Omega}_1 \). Assume that \( \partial \Omega_1 \) and \( \partial \Omega_2 \) satisfy Catlin's condition (P). Then the compactness estimate for \( (p, q) \)-forms with \( 0 < q < n - 1 \) holds for the \( \bar{\partial} \)-Neumann problem on \( \Omega \). This result implies that given a \( \bar{\partial} \)-closed \( (p, q) \)-form \( \alpha \) with \( 0 < q < n - 1 \), which is \( C^\infty \) on \( \overline{\Omega} \) and which is cohomologous to zero on \( \Omega \), the canonical solution \( u \) of the equation \( \bar{\partial}u = \alpha \) is smooth on \( \overline{\Omega} \).

1. Introduction

Let \( X \) be a complex manifold of dimension \( n \). Let \( \Omega \Subset X \) be an open submanifold with smooth boundary. Let \( \alpha \) be a \( \bar{\partial} \)-closed \( (p, q) \)-form, which is \( C^\infty \) on \( \overline{\Omega} \) and which is cohomologous to zero on \( \Omega \). The \( \bar{\partial} \)-Neumann problem is concerned with the existence and especially with the regularity of the solution \( u \) of the equation \( \bar{\partial}u = \alpha \), where \( u \) is orthogonal to the kernel of \( \partial \). This solution is called the canonical solution or Kohn's solution. By a theorem of Kohn and Nirenberg [8], it is known that the existence of a subelliptic estimate yields a positive answer to the question of local regularity. Catlin has proved that on a smoothly bounded pseudoconvex domain \( \Omega \) in \( \mathbb{C}^n \) a subelliptic estimate for the \( \bar{\partial} \)-Neumann problem holds in a neighborhood of \( p \in \partial\Omega \) iff \( \partial\Omega \) is of finite type at \( p \) [3]. For many applications, such as the boundary regularity of biholomorphic maps [1], it is sufficient to study the

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question of global regularity. Kohn has shown that when $\Omega$ is pseudoconvex, then there always is a smooth solution to $\bar{\partial} u = \alpha$ [9]. But in this result the solution $u$ is not necessarily the canonical solution.

A compactness estimate is said to hold for the $\bar{\partial}$-Neumann problem on $\Omega$ if for every $\epsilon > 0$ there is a $C(\epsilon) > 0$ such that

$$
\|\Phi\|^2 \leq \epsilon Q(\Phi, \Phi) + C(\epsilon)\|\Phi\|^2_{L^1}
$$

for all $\Phi \in \text{Dom}(\bar{\partial}) \cap \text{Dom}(\bar{\partial}^*)$. Here $Q(\Phi, \Psi)$ refers to the form $(\bar{\partial}\Phi, \bar{\partial}\Psi) + (\bar{\partial}^*\Phi, \bar{\partial}^*\Psi)$, and $\|\cdot\|^2_{L^1}$ refers to the Sobolev norm of order 1 for forms on $\Omega$.

We know that the property (1.1) is equivalent to the norm $Q$ being compact.

We define

$$
\mathcal{H}^{p,q} = \{ \Phi \in \text{Dom}(\bar{\partial}) \cap \text{Dom}(\bar{\partial}^*) \mid \bar{\partial}\Phi = 0 \text{ and } \bar{\partial}^*\Phi = 0 \}
$$

**Theorem 1.1 ([8]).** Let $\Omega \subseteq X$ be an open submanifold with smooth boundary. Let $m$ be a nonnegative integer and let $H^m(\Omega)$ be a Sobolev space of order $m$. Suppose that the compactness estimate (1.1) holds on $\Omega$. Suppose further that the $\bar{\partial}$-closed $(p,q)$-form $\alpha$ is in $H^m(\Omega)$ and $\alpha \perp \mathcal{H}^{p,q}$, then there is a constant $C_m$ so that the canonical solution $u$ of $\bar{\partial} u = \alpha$ with $u \perp \text{Ker}(\bar{\partial})$ satisfies

$$
\|u\|^2_m \leq C_m(\|\alpha\|^2_m + \|u\|^2).
$$

Since $C^\infty(\overline{\Omega}) = \cap_{m=0}^\infty H^m(\Omega)$, it follows that if $\alpha \in C^\infty_{(p,q)}(\overline{\Omega})$, then $u \in C^\infty_{(p,q-1)}(\overline{\Omega})$.

**Definition 1.2.** The boundary of a domain $\Omega$ satisfies property (P) if for every positive number $M$ there is a plurisubharmonic function $\lambda \in C^\infty(\overline{\Omega})$ with $0 \leq \lambda \leq 1$, such that for all $z \in \partial\Omega$,

$$
\sum_{j,k=1}^n \lambda_{jk}(z) t_j t_k \geq M |t|^2,
$$

where $\lambda_{jk}(z)$, $j, k = 1, \ldots, n$, is defined by $\partial \partial \lambda(z) = \sum_{j,k=1}^n \lambda_{jk}(z) \omega^j \wedge \overline{\omega}^k$ for an orthonormal basis $\omega^1, \ldots, \omega^n$ of $\Lambda^{1,0}$.

**Remark 1.9.** (1) Since the complex Hessian is preserved under biholomorphic mappings, the definition of property (P) does not depend on coordinate functions.

(2) Property (P) is known to hold for a large class of domains, including domains of finite type [3]. But even a domain with an infinitely flat spot, such as

$$
\Omega = \{ z \in \mathbb{C}^2 \mid |z_1|^2 + e^{-1/|z_1|^2} < \frac{1}{2} \}
$$

satisfies still property (P) [10].

In [2], Catlin showed that if $\Omega$ is a smoothly bounded pseudoconvex domain in $\mathbb{C}^n$ such that $\partial \Omega$ satisfies property (P), then the compactness estimate holds for the $\bar{\partial}$-Neumann problem on $\Omega$. In this paper, we consider the annulus $\Omega = \Omega_2 \setminus \overline{\Omega}_1$ between two pseudoconvex submanifolds $\Omega_1$ and $\Omega_2$ such that $\Omega_1 \subseteq \Omega_2$. We shall