EQUIVARIANT HARMONIC MAPS INTO THE SPHERE VIA ISOPARAMETRIC MAPS

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By using concrete isoparametric maps we obtain some new equivariant harmonic maps between spheres and solve equivariant boundary value problems for harmonic maps from unit open ball $B^{m+1}$ into $S^n$.

1. INTRODUCTION

In order to have harmonic maps into positively curved Riemannian manifolds one of the effective methods is the equivariant method with respect to natural geometric structures. In a previous paper [21] the author investigated equivariant harmonic maps in the framework of Riemannian submersions. The present one continues this study and supplies more examples and applications.

There are several definitions of isoparametric maps in the literature ([1],[5] and [20]). In this paper we adopt one in [20] which is adequate for constructing new harmonic maps. The isoparametric maps have been used to construct minimal submanifolds in $R^n$ and $S^n$ [4], [20]. We now show that they are also useful for general harmonic maps.

In Sec.2, we give the definition of the isoparametric maps and a basic property which enables us to use a reduction theorem in [21]. Then in Sec.3 we define an equivariant map with respect to isoparametric maps in the sphere and derive its harmonicity equation - a system of ODE. Under certain conditions we show the existence of its solutions. Thus, we obtain new harmonic maps between the spheres.

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By joining two harmonic polynomials R.T. Smith found many interesting harmonic maps between spheres [18]. His construction can also be reinterpreted as the equivariant harmonic maps with respect to isoparametric functions in the sphere. The present work can be viewed as a continuation of the earlier work.

In the last section we consider the equivariant boundary value problems for harmonic maps. The earlier fundamental results, due to R.S. Hamilton and Hildebrandt-Kaul-Widman, require that the images lie in the convex supporting neighbourhood [8], [14]. Later, in [15], [4] they consider rotationally symmetric harmonic maps from $B^m$ into $S^n$ whose boundary value just outside of the convex supporting neighbourhood. Recently, many works have been done on maps from $B^3$ into $S^2$ which are relevant to the study of liquid crystals [9], [10], [11], [12], [22]. Among them, D. Zhang obtained a regular axially symmetric harmonic extension of $B^3$ into $S^2$ for any given smooth axially symmetric boundary data which omits a neighbourhood of the south pole [22]. This restriction has been avoided by Hardt-Kinderlehrer-Lin [11].

It is reasonable to investigate the harmonic extensions from $B^{m+1}$ into $S^n$ for a given boundary data in a large image area. To simplify the equations we employ the equivariant technique. As is well known, there are only two different kinds of isoparametric hypersurfaces in the Euclidean space: umbilical ones and the generalized cylinder. They are corresponding to two kinds of reductions given by Jäger-Kaul [15] and Zhang [22], respectively. By putting the problem in this framework together with some essentially technical improvement we are able to generalize Zhang's result into the higher dimensional case.

In most previous works the various problems were reduced to a single ODE of second order. In Sec.3 we deal with a system of ODE of second order with two unknowns. Whereas, in Sec.4 we face to solve a second order PDE with two independent variables. Both cases are treated by the variational method in this paper.

2. ISOPARAMETRIC MAPS

In [20] isoparametric maps have been introduced as follows.

**Definition 2.1** A non-constant map $f$ from Riemannian manifold $M$ into $\mathbb{R}^n$ is called isoparametric, if for all $i, j = \ldots, n,$