ON SET THEORETIC COMPLETE INTERSECTIONS IN $\mathbb{P}^3_k$

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We prove that smooth monomial curves of degree greater than three are not set theoretic complete intersections on a wide class of surfaces, called bihomoegeneous surfaces.

0. Introduction

A well known and still open problem in algebraic geometry is whether every connected projective curve in $\mathbb{P}^3_k$ can be described as the intersection of two surfaces (set theoretic complete intersection). In 1882 L. Kronecker [4] proved that any space curve can be described as the intersection of four surfaces. And in 1960 M. Kneser [5] proved that three surfaces are enough.

It is a conjecture that one can find examples of curves that are not set theoretic complete intersections among monomial curves. A monomial curve in $\mathbb{P}^3_k$ is a curve parametrized by $(t^d, t^{a_1}u^{b_1}, t^{a_2}u^{b_2}, u^d)$ where $d, a_1, a_2, b_1, b_2$ are positive integers such that $a_1 + b_1 = a_2 + b_2 = d$. There is a large number of results in the literature concerning monomial curves. Among them R. Hartshorne [1] proves that, in positive characteristic, smooth monomial curves are set theoretic complete intersections and one of the surfaces is given by a binomial. Later on T. T. Moh [6] generalizes Hartshorne's results by proving that every monomial curve in $\mathbb{P}^3_k$ is set theoretic complete intersection on binomial hypersurfaces. In characteristic zero the problem is still open. L. Robbiano and G. Valla [7] prove that monomial curves in $\mathbb{P}^3_k$ which are arithmetically Cohen-
Macaulay are set theoretic complete intersections. And they also show that $C_4$, the smooth monomial curve of degree four is not set theoretic complete intersection on anyone of the three binomial surfaces $X_0^2X_2 - X_1^3$, $X_0X_3 - X_1X_2$ and $X_1^2 - X_2^3$. Recently, D. Jaffe’s work [2], [3] shows that smooth monomial curves of degree greater than three are not set theoretic complete intersections on surfaces with at most ordinary nodes as singularities or of degree at most three or cones.

The main result in this paper is that smooth monomial curves of degree greater than three are not set theoretic complete intersections on surfaces defined by bihomogeneous polynomials. A polynomial

$$f = \sum a_{n_0n_1n_2n_3}x_0^{n_0}x_1^{n_1}x_2^{n_2}x_3^{n_3} \in K[x_0, x_1, x_2, x_3]$$

is called bihomogeneous of type $(d, \alpha_1, \alpha_2)$ and degree $(a, b)$ if $\alpha_{n_0n_1n_2n_3} = 0$ for all $(n_0, n_1, n_2, n_3)$ with

$$n_0(d, 0) + n_1(\alpha_1, d - \alpha_1) + n_2(\alpha_2, d - \alpha_2) + n_3(0, d) \neq (a, b).$$

In particular binomial and trinomial surfaces passing through a monomial curve $C$ are always bihomogeneous. Note that bihomogeneous polynomials are always homogeneous and also that the ideal $I(C)$ of a monomial curve $C$ can be generated by bihomogeneous polynomials of the appropriate type $(d, \alpha_1, \alpha_2)$.

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2. Let $K$ be an algebraically closed field of characteristic zero and let $C$ be a smooth monomial curve, i.e. a curve embedded in $P_k$ with generic zero $(t^d, t^{d-1}u, tu^{d-1}, u^d)$ where $d$ is a positive integer and $d > 3$. The associated ideal $I(C)$ of $C$ is the kernel of $\phi: K[x_0, x_1, x_2, x_3] \to K[t, u]$ given by $\phi(x_0) = t^d$, $\phi(x_1) = t^{d-1}u$, $\phi(x_2) = tu^{d-1}$, $\phi(x_3) = u^d$.

In the sequel we are going to prove that $C$ is not set theoretic complete intersections on bihomogeneous surfaces. Let $F$ and $G$ be polynomials in $K[x_0, x_1, x_2, x_3]$ such that one of them is bihomogeneous, we are going to assume that $I(C) = \sqrt{(F, G)}$ and then arrive to a contradiction. Note that by Lemma 2.2 of [8] if one of the two surfaces is bihomogeneous then the other surface can be chosen to be bihomogeneous, so we can assume that $F$ and $G$ are bihomogeneous of the type $(d, d-1, 1)$. 

262