A NOTE ON THE HODGE INDEX THEOREM

Tie Luo

We prove that if a divisor $X$ is big and nef and $I(X^{(n-1)}, E) = 0$, where $E$ is an effective divisor and $n$ is the dimension of the variety $V$, then $h^0(X + E) = h^0(X)$. The technical tool is the generalized Hodge index theorem.

The purpose of this note is to generalize the Hodge index theorem from the surface case to higher dimensional algebraic varieties and give one application of the generalized form to the proof of an interesting theorem concerning the dimensions of certain linear systems (see Section 2 of [Luo]), which says that adding an effective divisor $E$ to a big and nef divisor $X$ has no effect on the dimension of the global sections, provided $E$ intersects $X$ trivially. The proof in [Luo] uses some delicate analysis on the fixed components of high multiples of $X$.

The following notations are oftenly used:
- $I(-,...,-)$: The intersection number of certain divisors.
- A Q-divisor is a rational combination of some Weil divisors.
- A divisor is called nef if it intersects any curve non-negatively.
- A divisor is called big if its self-intersection number is positive.
- $h^0(D)$: The dimension of $H^0(V, O(D))$ for the cartier divisor $D$ on the variety.

Let us first recall the Hodge index theorem in the surface case (see [Zariski]):

Let $S$ be a projective non-singular algebraic surface over $C$. $X$ is a big divisor on $S$. If $E$ is a divisor such that $I(X, E) = 0$, then $E(2) \leq 0$ and "=" holds if and only if $E$ is numerically equivalent to zero.
We are going to generalize it to the following

**Theorem 1:** Let V be a projective non-singular algebraic variety of dimension n over C. X is a nef and big divisor on V. If E is a divisor such that I(X(n-1),E)=0, then I(X(n-2),E(2))=0 and "-" holds if and only if X(n-2).E is numerically equivalent to zero.

**Remark:** There are examples for which I(X(n-1),E)=0 and I(X(n-2),E(2))=0. Of course in this case X(n-2).E is numerically trivial.

Before the proof of the theorem, we need the following lemma which can be easily proved.

**Lemma:** The theorem is true if X is an ample Q-divisor and E is a Q-divisor.

**Proof of the theorem:** Let us prove the first part of the theorem, i.e. under the assumptions we have I(X(n-2),E(2))=0.

We can assume X(n-2).E is not numerically trivial. Let us perturb X and E by a ample divisor H such that I(X(n-2),H,E)=0 and define

\[ X_s = X + sH \quad E_t = E + tH \]

where s and t are rational numbers. Thus \( X_s, E_t \in \text{Pic}(V) \otimes \mathbb{Q} \)

Now consider I(X_s(n-1),E_t) and I(X_s(n-2),E_t(2)).

\[ I(X_s(n-1),E_t) = I((X+sH)(n-1),(E+tH)) = I(X(n-1),E) + (n-1)sI(X(n-2),H,E) + \text{higher degree terms in } s + t \]

\[ I(X(n-1),E) = 0 \text{ by assumption. We may choose } s \text{ to be a very small positive rational number and solve the equation} \]

\[ I(X_s(n-1),E_t) = 0 \]

to get a equally small rational number \( |t| \).

When that is so,

\[ I(X_s(n-2),E_t(2)) = I((X+sH)(n-2),(E+tH)(2)) = I(X(n-2),E(2)) + \text{terms with } t \text{ or } s \text{ involved.} \]