Partial Regularity for Certain Classes of Polyconvex Functionals Related to Nonlinear Elasticity

Martin Fuchs, Jürgen Reuling

Fachbereich 9 Mathematik der Universität des Saarlandes, Postfach 151150, D-66041 Saarbrücken, Germany

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Consider the variational integral

$$J(u) := \int_{\Omega} |\nabla u|^p + H(\det \nabla u) \, dx$$

where $\Omega \subset \mathbb{R}^n$ and $p \geq n \geq 2$. $H : (0, \infty) \to [0, \infty)$ is a smooth convex function such that $\lim_{t \to 0} H(t) = 0$. We approximate $J$ by a sequence of regularized functionals $J_{\delta}$ whose minimizers converge strongly to an $J$-minimizing function and prove partial regularity results for $J_{\delta}$-minimizers.

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0 Introduction

We study a special class of polyconvex variational integrals which are related to nonlinear elasticity. Our main purpose is to illustrate some ideas which might lead to partial regularity of minimizers for stored energies studied in the papers of John Ball (see [3],[4]). To be precise consider a bounded open set $\Omega$ in $\mathbb{R}^n$ and a real number $p \geq n$. We require $n \geq 2$ and $p > 2$ — the case $n = p = 2$
has been treated in the paper [6]. Suppose further that we are given a function 
    \( u_0 \in H^{1,p}(\Omega, \mathbb{R}^n) \) such that
    \[ \tau \leq \det \nabla u_0(x) \leq \frac{1}{\tau} \quad \text{a.e. on } \Omega \]
for some \( \tau \in (0, 1) \). Then we look at the variational problem
\[
J(u) := \int_{\Omega} |\nabla u|^p + H(\det \nabla u) \, dx
\]
with \( H : (0, \infty) \to [0, \infty) \) of class \( C^2 \), strictly convex and with the property
\[
\lim_{t \to 0} H(t) = \infty.
\]
Integrands of this type occur as stored energy densities for certain models from
nonlinear elasticity (see Ball [3],[2] and Ogden [9]) and from the work of Ball
[3],[4] or Müller [8] we deduce that problem (0.1) has at least one solution \( u \in C \).
Up to now nothing is known about the regularity properties of minimizers \( u \) but
the results described below give rise to the following

**CONJECTURE:** There is an open subset \( \Omega_0 \) of \( \Omega \) whose complement has
vanishing Lebesgue measure such that \( u \in C^{1,\alpha}(\Omega_0) \) for any \( 0 < \alpha < 1 \). Moreover,
\( x_0 \in \Omega_0 \) if and only if the following conditions hold:

a) \( x_0 \) is a Lebesgue point for \( \nabla u \)
b) \( \det \nabla u(x_0) > 0 \)
c) \[
\int_{B_{r}(x_0)} |\nabla u - (\nabla u)_{x_0,r}|^p \, dx \to 0 \quad \text{as } r \downarrow 0.
\]
Here and in the sequel we use the symbol \((f)_{x_0,r}\) to denote the mean value
\[
\int_{B_{r}(x_0)} f \, dx
\]
of the function \( f \).

As a first approach towards this conjecture we consider the case
\[
\lim_{t \to \infty} H'(t) = \infty
\]
and replace (0.1) by a sequence of more regular variational problems
\[
J_\delta(v) := \int_{\Omega} |\nabla v|^p + h_\delta(\det \nabla v) \, dx \to \min \text{ in } C
\]
where for \( 0 < \delta < \tau \)
\[
h_\delta(t) := \begin{cases}
H'(\delta)(t - \delta) + H(\delta) & , \quad t \leq \delta \\
H(t) & , \quad \delta \leq t \leq \delta^{-1} \\
H'(1/\delta)(t - 1/\delta) + H(1/\delta) & , \quad t \geq \delta^{-1}
\end{cases}
\]