A DIRECT PROOF FOR LOWER SEMICONTINUITY OF POLYCONVEX FUNCTIONALS

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Lower semicontinuity for polyconvex functionals of the form \( \int_{\Omega} g(\det Du)dx \), with respect to sequences of functions from \( W^{1,n}(\Omega; \mathbb{R}^n) \) which converge in \( L^1(\Omega; \mathbb{R}^n) \) and are uniformly bounded in \( W^{1,n-1}(\Omega; \mathbb{R}^n) \), is proved. This was first established in [5] using results from [1] on Cartesian Currents. We give a simple direct proof which does not involve currents. We also show how the method extends to prove natural, essentially optimal, generalizations of these results.


Key Words: polyconvex, lower semicontinuity.

1. Introduction

We first consider functionals of the form

\[ \int_{\Omega} g(\det Du)dx \]

for functions \( u \in C^1(\Omega; \mathbb{R}^n) \), or more generally \( u \in W^{1,n}(\Omega; \mathbb{R}^n) \). Here \( \Omega \subset \mathbb{R}^n \) is a bounded open set and \( g: \mathbb{R} \to \mathbb{R} \) is convex and satisfies

\[ g(t) \geq a|t| \]

for some \( a > 0 \).

It is well known that if \( u, u_h \in W^{1,n}(\Omega; \mathbb{R}^n) \) and \( u_h \rightharpoonup u \) weakly in \( W^{1,n} \) then

\[ \int_{\Omega} g(\det Du)dx \leq \liminf_{h} \int_{\Omega} g(\det Du_h)dx \]

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Extending previous results proved by Marcellini in [14], it was later shown in [4] that inequality (1.1) still holds if we assume that the sequence $u_h$ converges to $u$ weakly in $W^{1,p}$ for $p > n - 1$. The limit case $p = n - 1$ has been considered in [5] (and in a particular case also in [12]), where it was proved, using results from [1] on Cartesian Currents, that (1.1) remains true if $u_h, u \in W^{1,n}(\Omega; \mathbb{R}^n)$, $u_h \rightharpoonup u$ in $L^1(\Omega; \mathbb{R}^n)$, and $\int_\Omega |D u_h|^{n-1} \leq M < \infty$ uniformly in $h$. We remark that this result is optimal with respect to Sobolev norms, in that a recent counterexample of Maly [12] shows that lower semicontinuity does not hold for weak $W^{1,p}(\Omega; \mathbb{R}^n)$ convergence if $p < n - 1$.

The result (1.1) involves only the classical notion of determinant, and in particular involves neither the weak notion of determinant due to Ball [2] nor the notions of Cartesian Currents and their boundaries as developed in [10,11] and elsewhere. It is thus natural to look for a proof which avoids the use of currents, and in particular the deep compactness and lower semicontinuity results available in this setting.

In Section 2 we give such a direct proof, using in particular the truncation and projection argument of [8]. More precisely, the sequence $(u_h)$ from (1.1) is replaced by another sequence $v_h \rightharpoonup u$ in $L^1$. It is straightforward to show that

$$\int_\Omega g(\det Du) dx \leq \liminf_h \int_\Omega g(\det D v_h) dx,$$

see Lemma 2.2. Thus one needs to establish that

$$\liminf_h \int_\Omega g(\det D v_h) dx \leq \liminf_h \int_\Omega g(\det D u_h) dx.$$

This last inequality follows from an argument which involves not only $\det D u_h$ (and $\det D v_h$) but also all the minors of smaller degree. The main point here is that minors are well-behaved under the truncation and projection operation used in Proposition 2.5, although the truncated functions $(v_h)$ need not be uniformly bounded in $W^{1,n-1}(\Omega; \mathbb{R}^n)$.

Under the weaker assumption that $u_h \rightharpoonup u$ in $L^1$ and the minors of $D u_h$ are uniformly bounded in $L^1$, the same argument also establishes analogous lower semicontinuity results for general polyconvex integrands depending only on the gradient. See Theorem 2.6.

In Section 3 we obtain analogous results for integrands of the form $f(x,u,Du)$. These results were obtained in [5] and [1] using the theory of currents. We avoid this machinery by means of the blow-up argument used in [8] in order to reduce the problem to the previous case of no $x$ or $u$ dependence.

Finally, we extend these results to the natural setting of sequences from the class

$$\mathcal{A}_q(\Omega; \mathbb{R}^N) = \{ u \in W^{1,q}(\Omega; \mathbb{R}^N) : \mathcal{M}_{k-1}(D u) \in L^{q/(q-1)}(\Omega) \}$$

for $\Omega \subset \mathbb{R}^n$ a bounded open set, $k - 1 \leq q \leq k$, $k = \min\{n,N\} > 1$. Here $\mathcal{M}_{k-1}(D u)$ is the vector of $k-1$ minors. This class has been recently considered in [16], [15] and [11]. The main point in extending the earlier proof to this setting is an extension of the co-area formula from Lipschitz to $W^{1,1}$ functions.