Nonlinear and buckling analysis of continuous bars lying on elastic supports, based on the theory of elastica

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Summary: Based on the theory of elastica, a parametric solution for the problem of nonlinear and buckling analysis of continuous bars on elastic supports was presented. This problem, for the case of a continuous bar on rigid supports not on the same level, led to a nonlinear system of $3(q - 1)$-equations with $4(q - 1)$-unknowns, which was further enriched by $(q - 2)$ three-moment nonlinear equations based on convenient compatibility conditions. Furthermore, for the case of elastic supports, $q$-additional nonlinear equations were formulated including the deflections and the spring constants of the supports. Applications of the proposed methodology in solving the previous systems were examined and several numerical results were derived.

Eine auf der Theorie der Elastica basierende nichtlineare Analyse des Knickens von kontinuierlichen Balken auf elastischen Auflägern


1 Introduction

The second order theory of linear buckling for continuous beams subjected to longitudinal compressive forces has received considerable attention in recent years. A thorough up to that day examination of this problem was presented by Love [1] and Timoshenko, Gere [2]. The researchers consider the bar lying on rigid or elastic supports. Also, in these references the calculation of the critical compressive force was based on the three-moment equation.

On the other hand, the postbuckling behavior of an elastic column subjected to end compressive forces dates back to Euler [1, 2]. In [3] a parametric solution of the problem of elastica of a simply supported bar subjected to a compressive force and a bending moment is given in terms of tabulated elliptic integrals of the first and the second kind. Finally, the closed-form solutions of the strongly nonlinear differential equations concerning the nonlinear and buckling analysis of cantilevers, subjected to a general terminal co-planar loading, have been given in [4] and [5], taking into account the influence of transverse shear deformation.

In the present investigation the nonlinear and buckling analysis of an elastic continuous bar on $q$-elastic supports, subjected to terminal compressive forces, is presented. The methodology developed was based on the third-order theory, i.e. on the exact differential equation of the deflection curve. First, two arbitrary consecutive spans of the bar, lying on rigid supports not on the same level, are analyzed to equivalent members. These two members are subjected to terminal bending moments, shear and axial forces. Supposing that the compressive forces and the flexural rigidities may vary from one span to the next, the strongly nonlinear equilibrium
differential equation of each deformed member is formulated and the closed-form solution of this equation (expressed by elliptic integrals) is derived. In the sequel, through convenient boundary and compatibility conditions a nonlinear system of \(3(q - 1)\)-equations with \(4(q - 1)\)-unknowns is formulated, which is further enriched by a set of \((q - 2)\) three-moment equations. If the supports are elastic, an additional nonlinear system of \(q\)-equations with equal unknowns is formulated, including the deflections and the spring constants of the supports. By now, selecting values for the slope of the deflection of the first support, as well as for the elliptic integral (for the first span) appearing in the derived equations, a parametric (closed-form) solution of the previous systems is achieved. This solution is illustrated in the applications being examined. These applications concern:

i) The buckling of a continuous bar of equal lengths on three supports in which the middle support is elastic, and

ii) The buckling of a continuous bar of unequal lengths on three rigid supports.

The proposed herein methodology is convenient for applications to aerospace structural problems, as well as for engineering structures where large elastic deformations are required.

2 Analysis

Consider a slender continuous straight bar on rigid supports not on the same level, which is subjected to terminal axial compressive forces. Let \(1; 2; \ldots; n; \ldots; q\) denote the consecutive supports; \(M_2; M_3; \ldots; M_n; \ldots; M_{q-1}\) the corresponding statically indeterminate bending moments; \(l_1; l_2; \ldots; l_n; \ldots; l_{q-1}\) the span-lengths; and \(\theta_1; \theta_2; \ldots; \theta_n; \ldots; \theta_{q-1}\) the corresponding slopes of the deflection curve for each span. It must be noticed here that the previous angles of rotation \(\theta\) are taken as positive when they are in the same directions as the positive bending moments (Fig. 1). We denote by \(\theta_{n-1}, \theta_{n+1}\) the angles of rotation of the left or the right support of the \(n\)-span respectively; by \(Q_{n-1}, Q_{n+1}\) the corresponding support-reactions, and, finally, by \(\delta_n\) the ordinate of the \(n\)-support.

It is further assumed that the compressive force \(P\) and the flexural rigidity \(EJ\) may vary from one span to the next, but within each span these quantities are taken as constant. In calculating the critical buckling loads for the previous bar, based on the theory of elastica, we must, in principle, analyze any two consecutive spans \((l_{n-1}; l_n)\), shown in Fig. 2, to equivalent members \(l_{n-1}\) and \(l_n\). Each of these members can be considered in general as a simply supported bar, which, because of large deformations, is held on one or two movable ends, and it is loaded by terminal moments and axial and shear forces. Finally, the angles \(\beta_{n-1}; \beta_n\), shown in Figs. 1 and 2, are defined by the equations

\[
\cos \beta_{n-1} = (\delta_n - \delta_{n-1})/l_{n-1}; \quad \cos \beta_n = (\delta_{n+1} - \delta_n)/l_n
\]

respectively.

Fig. 1. Geometry and sign convention of two consecutive spans of a continuous bar on rigid supports not on the same level