THE FINITENESS OF A BASE OF IDENTITIES
FOR FIVE-ELEMENT MONOIDS

A.V. Tishchenko
Communicated by L.N. Shevrin

P. Perkins [6] has given an example of a monoid, i.e. a semigroup with identity, of order six whose identities are not finitely based. He also showed that any semigroup of order three had finitely based identities. The question whether a semigroup of order four and five has a finite base of identities was raised in [8]. A.D. Bolbot [2] announced that identities of any semigroup of order four were finitely based. One can find a proof in [3]. C.C. Edmunds [4] has proved that identities of any monoid with zero of order five are finitely based.

The main result of the present paper is the following theorem announced in [9].

THEOREM. Any monoid of order five has a finite base of identities.

In order to prove this theorem we shall use the following well known results: 1) identities of a commutative semigroup have a finite base [6]; 2) identities of a band are finitely based [1]; 3) a list of all semigroups of order less than five [5].

Below we shall not distinguish semigroups which are isomorphic or antiisomorphic.

The following well known simple fact is useful for us.

LEMA 1. A finite (even periodic) monoid S can be
represented as a union of a group $G$ of all (left) invertible elements in $S$ and the unique maximal ideal $T$ in $S$ such that $G \cap T = \emptyset$. Hence, if $T$ is not empty then $S$ is a semilattice of the ideal $T$ and the group $G$.

In order to obtain multiplication tables of all five-element monoids we shall vary the order of the group $G$ in lemma 1 from 1 to 5.

1. PRELIMINARIES

Let $X = \{x, x_1, x_2, \ldots, y, y_1, y_2, \ldots, z, z_1, z_2, \ldots\}$ be a countable alphabet. For a word $A$ in alphabet $X$ let $\ell(A)$ denote the length of $A$, $\ell_x(A)$ denote the number of occurrences of variable $x$ in $A$, $c(A)$ the set of variables which occur at least once in $A$, $c_k(A)$ the set of all variables having exactly $k$ occurrences in $A$. If $A$ and $B$ coincide graphically then we shall write $A \equiv B$.

An identity $A = B$ is called homogeneous if $c(A) = c(B)$. A system of identities $\beta$ is called closed under deleting (see [6]) if every identity of $\beta$ is homogeneous and every identity obtained from $A = B$ by deleting all occurrences of any variable $x$ again belongs $\beta$. It is easy to see that a set of all identities of a semigroup with identity element which is not a group is closed under deleting.

A semigroup $S$ is called an ordinal sum of $T$ and $G$ if $S$ is a semilattice of $T$ and $G$ and $tg = gt = t$ for each $t \in T$ and $g \in G$. We also say that $S$ is an ordinal extension of $T$.

In order to find a base of identities of some semigroups we need two following obvious lemmas.

**LEMMA 2.** If a semigroup $S$ contains an ideal $T$ and a subsemigroup $G$ such that $T \cap G = \emptyset$ then each identity of $S$ is homogeneous.

**LEMMA 3.** If $S$ contains a left-zero subsemigroup then for each identity $A = B$ of $S$ the words $A$ and $B$