GROUP IDEALS IN A SEMIGROUP OF MEASURES

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Abstract: In this paper we introduce the notion of a group ideal in a semigroup. We shall prove that all group ideals of a compact affine semigroup are convex and dense. This generalizes many results in the literature concerning ideals in semigroups.

Let $S$ be a compact semigroup, and let $P(S)$ denote the set of all probability measures on $S$. It is well known that $P(S)$ is a compact affine semigroup under convolution and the weak star topology. The properties of ideals in $P(S)$ were first studied by Cohen and Collins in [11], where they noticed that all minimal left (right) ideals of $P(S)$ are convex. Recently, Chow has found a large variety of classes of ideals which are convex and dense in $P(S)$, e.g., open prime ideals, open semiprime ideals, maximal ideals, nilpotent radicals, etc. (see [2]-[9]). We introduce the notion of a group ideal in a compact semigroup, and then show that all group ideals $\Delta$ of $P(S)$ are convex and dense in $P(S)$ and $S(\Delta) = S$. This result includes all the results of Chow as immediate corollaries.

1. Definition. An ideal $I$ of a semigroup $S$ is a group ideal if, whenever $x \not\in I$, then $J(x)$, the principal ideal generated by $x$, contains a group $G$ which is disjoint from $I$. $I$ is called topologically semiprime if, whenever $x \not\in I$, then $\Gamma(x) \cap I = \emptyset$, where $\Gamma(x) = \{ x^n : n \geq 1 \}$.

If $S$ is a compact semigroup, then it is easy to see that open semiprime ideals are topologically semiprime, and that topologically semiprime ideals are group ideals. However, the converse of either statement is false. It is also clear in the case of a compact semigroup that algebraic radicals of open ideals (respectively, the nil radical if $S$ has a zero, the topological radical of any ideal if $S$
is commutative) are all topologically semiprime.

2. **Proposition.** The intersection of all open prime ideals in a compact semigroup $S$ is a group ideal.

Proof. Let $P$ be an open prime ideal of $S$. If $x \notin P$, then $P^*(x) \cap P = \emptyset$ as $P$ is open. Since $S$ is compact, there is an idempotent $e$ in $P^*(x)$ (see 1.1.10 of [16]). Hence the maximal subgroup $H(e)$ is in $J(x)$ and is disjoint from $P$. Thus $P$, and hence the intersection of all open prime ideals of $S$ are group ideals.

Note that the intersection of any family of ideals of any of the types mentioned in the paragraph preceding the proposition is again an ideal of the same type, and so must also be a group ideal.

3. **Definition.** Given a non-empty subset $\Delta$ of $P(S)$, the set $V(\Delta) = \{tm + (1-t)v : \mu \in \Delta, v \in P(S), 0 < t < 1\}$ is called the open convex hull of $\Delta$.

The straightforward proofs of the following facts can be found e.g., in [2]: If $I$ is an ideal of $S$, then $\widetilde{I} = \{\mu \in P(S) : \text{supp } \mu \cap I \neq \emptyset\}$ is an ideal of $P(S)$. For an ideal $\Delta$ of $P(S)$, the set $S(\Delta) = \cup \{\text{supp } \mu : \mu \in \Delta\}$ is an ideal of $S$, and $\widetilde{I} \supset \Delta$. $V(\Delta)$ is dense in $P(S)$, and $S(V(\Delta)) = S$.

4. **Lemma.** If $\Delta$ is a group ideal of $P(S)$, then $\Delta = V(\Delta)$. More generally, $V(I) = I$ for any group ideal $I$ of a compact affine semigroup.

Proof. We prove the result for the case $P(S)$. Let $\mu_0 \in P(S) \setminus \Delta$. Then, since $\Delta$ is a group ideal of $P(S)$, there is a group $G \leq J(\mu_0)$ with $G \cap \Delta = \emptyset$. Let $\tau$ be the identity of $G$; then $\tau$ is an idempotent measure on $S$, and so $\tau^2 = \tau \notin \Delta$. If $\tau \in V(\Delta)$, then $\tau = tm + (1-t)v$ for some $\mu \in \Delta$ and some $v \in P(S), 0 < t < 1$. Hence $\text{supp } \mu \subset \text{supp } \tau$. This, together with Lemma 3 of [17] implies that $tm = \tau \in \Delta$, a contradiction. Thus $\tau \notin V(\Delta)$. But then $\mu_0 \notin V(\Delta)$, for $\mu_0 \in V(\Delta)$ implies that $\tau \in J(\mu_0) \subset V(\Delta)$, as $V(\Delta)$ is an ideal. Thus $V(\Delta) \subset \Delta$, which implies the result.

Our main theorem now follows from Lemma 4 and Proposition 2:

5. **Main Theorem.** If $\Delta$ is a group ideal of $P(S)$, then $\Delta$ is convex and dense.