BANDS ON TREES

by

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1. INTRODUCTION

Any metric tree with a finite number of end points can serve as the underlying space for a topological band (= idempotent semigroup) with zero. One merely takes the cartesian product of a finite band with at least as many points as the number of end points together with a "min" thread; then, divides out the appropriate congruence. Indeed, with a little care for the topological considerations, a similar process will work for trees with an infinite number of end points and even for many generalized trees. See [2].

The purpose here, however, is to consider the inverse question; namely, must every band with zero on a finite tree be of this sort? More precisely: Given a band with zero on a finite tree, S, does there exist a finite band, D, such that S is the continuous homomorphic image of D × I (where I is the unit interval with "min" multiplication)?

The answer is affirmative if S has an identity element and negative if no identity is required. An example is given to illustrate this.

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2. BANDS WITH IDENTITY

Throughout this section S will stand for a topological band with zero and identity whose underlying space is a tree with a finite
number of end points. We list here several well-known properties of
trees and semigroups which we need in the sequel.

The identity element which we designate by the letter \( u \) must
be an end point. See e.g. [3]. The zero element, \( O \), may be an end
point, a branch point, or neither. Any two distinct points \( p \), and
\( q \) have a unique arc connecting them. This arc is designated as
\([p,q]\). We say that \( q < p \) if \( q \neq p \) and \( q \in [p,O] \).

**Lemma 1.** The arc \([p,O]\) is a min thread.

**Proof.** Since multiplication is continuous and \( p \) is idempotent,
\( p \cdot [p,O] \) is a connected set containing \( p \) and \( O \) and, therefore
\([p,O]\). Hence every point in \([p,O]\) can be written as \( px \) for some
\( x \). So \( p \) is a left identity for \([p,O]\), and since a similar argu-
ments shows \( p \) is also a right identity, \( p \) is an identity for
\([p,O]\). This actually proves the lemma for we have shown that any
point is an identity for anything less than it.

**Lemma 2.** If \( x \in [u,O] \), then \( xp = px \) for all \( p \in S \).

**Proof.** We show first that \( p \cdot [u,O] = [p,O] = [u,O]p \). Since
\( p[u,O] \) contains \( pu = p \) and \( pO = 0 \) it must contain \([p,O]\). If
\( y = pt \in p[u,O] \), then the possibilities are i) \( y \in [p,O] \), ii)
\( p \in [y,O] \), iii) \( 0 \in [p,y] \) or iv) \( \exists b \), a branch point such that
\( b \in [p,O] \cap [y,O] \).

If \( p \in [y,O] \), by Lemma 1, \( p = py = p(pt) = pt = y \).
If \( 0 \in [p,y] \), \( \exists s > t \) such that \( ps = 0 \) and \( y = pt = pst =
Ot = 0 \).
If iv), \( \exists r \) and \( s \) such that \( r < t < s \) and \( b = pr = ps \).
Therefore \( y = pt = p(st) = (ps)t = (pr)t = p(rt) = pr =
b \in [p,O] \). So in all cases \( y \in [p,O] \) and \( p \cdot [u,O] = [p,O] \),
and a similar argument can be made for \( p \) on the right. Now
we are able to say that \( px = (px)p = p(xp) = xp \).

**Remark:** This lemma may easily be strengthened to say that if
\( x \in [e,O] \), \( y \in [f,O] \) and \( fe = ef \), then \( xy = yx \), but we will need
only the weaker form.