THE IDEAL STRUCTURE OF $X^X$

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1. INTRODUCTION.

Given a Hausdorff topological space $X$, the set $X^X$ of functions from $X$ to $X$ with the product topology is a right topological semigroup under composition. (By "right topological" we mean that, for each $f \in X^X$, the function $\rho_f$ defined by $\rho_f(g) = gf$ is continuous.) The spaces $X^X$ form probably the simplest class of right topological semigroups that are not semitopological. (The function $\lambda_f: g \mapsto fg$ is continuous if and only if $f$ is continuous.) The spaces $X^X$ also arise naturally in topological dynamics; see [1; p. 15] or [2; p. 9].

In this paper we study the ideal structure of $X^X$ in terms of certain closure operators on $\Pi(X)$, the set of partitions of $X$, and on $P_+(X)$, the set of non-empty subsets of $X$. We succeed in describing the right ideals, closed right ideals, minimal right ideals, left ideals, closed left ideals, and the (unique) minimal left ideal of $X^X$. In particular, we show that the closure of each left ideal in $X^X$ is again a left ideal. (In any right topological semigroup the closure of a right ideal is a right ideal. Example V.I.I. of [1], due to Ruppert in [3], shows that the closure of a left ideal need not generally be a left ideal.) Another consequence of our description of

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the left ideals is the surprising fact that the closure of a left ideal in $X^X$ does not depend on the topology on $X$.

2. SOME CLOSURE OPERATORS

We denote by $\Pi(X)$ the set of partitions of $X$ and by $P_+(X)$ the set of non-empty subsets of $X$. We order $P_+(X)$ by inclusion and $\Pi(X)$ by refinement. Thus, if $A, B \in \Pi(X)$, we write $A \leq B$ if $B$ is a refinement of $A$, that is, if, for each $A \in B$, there exists $A \in A$ such that $B \subseteq A$. If $A \in \Pi(X)$ and $F \subseteq X$, we denote by $A \vert_F$ the partition of $F$ induced by $A$. That is, $A \vert_F = \{A \cap F: A \in A\} \setminus \{\emptyset\}$. We denote by $N_x$ the set of open neighborhoods of $x$ in $X$.

2.1 DEFINITION. Let $A \subseteq P_+(X)$ and let $F \subseteq \Pi(X)$.

(a) $A^a = \{B \in P_+(X): B \subseteq A$ for some $A \in A\}$.

(b) $A^b = \{B \in P_+(X):$ for each finite non-empty subset $F$ of $B$ and each $\tau \in \times_{x \in F} N_x$, there exists $A \in A$ such that for all $x \in F$, $A \cap \tau(x) \neq \emptyset\}$.

(c) $A^c = \{B \in \Pi(X): B \subseteq A$ for some $A \in F\}$.

(d) $A^d = \{B \in \Pi(X):$ for each finite subset $F$ of $X$ there exists $A \in F$ such that $B \vert_F \leq A \vert_F\}$.

2.2 LEMMA. Each of $a, b, c,$ and $d$ is a closure operator. The topology induced by $b$ on $P_+(X)$ is coarser than that induced by $a$. The topology induced by $d$ on $\Pi(X)$ is coarser than that induced by $c$. The topologies induced by $a, c,$ and $d$ are $T_0$ but not (provided $|X| \geq 2$) $T_1$.

Proof. The proofs that $a$ and $c$ are closure operators are routine as are the proofs of the last three sentences of the lemma. There are only two points at which any difficulty arises in showing that $b$ is a closure operator. These are that $A^b \subseteq A^b$ and that $(A \cup B)^b \subseteq A^b \cup B^b$ for $A, B \in P_+(X)$. For the first, let $B \in A^b$. To see that $B \in A^b$ let $F$ be a finite non-empty subset of $B$ and let $\tau \in \times_{x \in F} N_x$. We may presume, since $X$ is Hausdorff, that $\tau(x) \cap \tau(y) = \emptyset$ for distinct $x$ and $y$ in $F$. Pick $C \in A^b$ such that, for all $x \in F$, $C \cap \tau(x) \neq \emptyset$. For $x \in F$, pick $y(x) \in C \cap \tau(x)$ and let $G = \{y(x): x \in F\}$. For $y \in C$, let $y(y) = \tau(x)$ where $x$ is the (unique) member of $F$ such that $y = y(x)$. Pick $A \in A$ such that, for all $y \in G$,