RESEARCH ARTICLE

Partial Orders in Semigroups and Semirings
of Right Quotients

Udo Hebisch
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1. Introduction

In [1] A. A. Albert considered the following question: Let \((S, +, \cdot, \leq)\) be a totally ordered ring and \((T, +, \cdot) = Q_r(S, S \setminus \{0\}) = Q_l(S, S \setminus \{0\})\) a field of right as well as left quotients of \((S, +, \cdot)\) in the sense of O. Ore. Does there exist an extension \(\leq^T\) of \(\leq\) on \(T\) such that \((T, +, \cdot, \leq^T)\) is also a totally ordered ring?

Then he showed that such an extension always exists. From Theorem 3 in Chapter VI of [3] one obtains that this is true even for every semiring \((T, +, \cdot) = Q_r(S, \Sigma) = Q_l(S, \Sigma)\) of (right and left) quotients of \((S, +, \cdot)\), and that \(\leq^T\) is uniquely determined by \(\leq\). Moreover, from Theorem 4 in Chapter X of [3] follows the corresponding statement for semigroups \((T, \cdot) = Q_r(S, \Sigma) = Q_l(S, \Sigma)\) of (right and left) quotients of a totally ordered semigroup \((S, \cdot, \leq)\).

Meanwhile these results have been generalized in several directions. For example, in [7] the above problem was investigated in the case of partially ordered commutative semigroups \((S, \cdot, \leq)\), and in [6] and [4] the more general case of semigroups \((T, \cdot)\) of right quotients of \((S, \cdot)\) was considered. Finally, in [5] totally ordered semirings \((S, +, \cdot, \leq)\) and their semirings of right quotients were considered, but both operations of \((S, +, \cdot)\) were assumed to be cancellative.

In this paper we will treat this question for partially ordered semirings and their semirings of right quotients in such a generality that we will cover all the cases mentioned above. For this purpose we investigate the same problem at first for arbitrary partial orders \(\leq\) on a semigroup \((S, \cdot)\), regardless of whether \((S, \cdot, \leq)\) is a partially ordered semigroup; i.e., we will not assume \(M'(\leq) = S\) for the set of all monotone elements (cf. Definition 2.1). Instead of this, we will merely assume that the denominators of the elements in a semigroup or semiring of quotients can be chosen to be monotone. In Thm 4.1 we obtain necessary and sufficient conditions (4.1) and (4.2) for the existence of a strict extension \(\leq^T\) of a partial order \(\leq'\) on \((S, \cdot)\), and we show that \(\leq^T\) is uniquely determined by \(\leq'\). In Remark 4.2 and Theorem 4.4 we give sufficient conditions to get an internal extension \(\leq'\) of an arbitrary partial order \(\leq\) on \((S, \cdot)\) such that (4.1) and (4.2) are satisfied. Applying these results to the semiring case, we obtain corresponding results in Theorems 5.1 and 5.2.

2. Partial Orders in Semigroups and Semirings

**Definition 2.1.** For every semigroup \((S, \cdot)\) and every partial order \(\leq\) on \(S\) we call \((S, \cdot, \leq)\) a generalized partially ordered (g. p. o.) semigroup, and a generalized totally ordered (g. t. o.) semigroup if \(\leq\) is even a total order. In every g. p. o. semigroup \((S, \cdot, \leq)\) we define

\[ M'_i(\leq) = \{ m \in S \mid a < b \implies ma \leq mb \text{ for all } a, b \in S \}, \]
the set of all left monotone elements. By left-right duality we get $M_r(\leq)$, the set of all right monotone elements, and finally the set of all monotone elements $M(\leq) = M_l(\leq) \cap M_r(\leq)$ of $(S, \cdot, \leq)$. As usual (cf. e. g. [3]), $(S, \cdot, \leq)$ is called a partially ordered (p. o.) semigroup if $M(\leq) = S$ holds. (In [6] g. p. o. semigroups $(S, \cdot, \leq)$ satisfying $M_r(\leq) = S$ are called partially right ordered semigroups, and in [2] totally right ordered groups have been investigated.) Moreover, we denote by

$$P_l(\leq) = \{p \in S \mid a \leq pa \text{ for all } a \in S\}$$

the set of all left positive elements, by $P_r(\leq)$ its dual, and finally by $P(\leq) = P_l(\leq) \cap P_r(\leq)$ the set of all positive elements of $(S, \cdot, \leq)$. (Obviously, all these subsets of $(S, \cdot)$ are subsemigroups if not empty.)

By a semiring we mean as usual an algebra $(S, +, \cdot)$ such that $(S, +)$ and $(S, \cdot)$ are semigroups for which multiplication distributes over addition as for rings (cf. e. g. [9]). If $(S, \cdot)$ has a neutral element $e$, it is called the identity, and if $(S, +)$ has a neutral element $o$, it is called the zero of the semiring $(S, +, \cdot)$. If additionally $oa = ao = o$ holds for all $a \in S$, $o$ is an absorbing zero of $(S, +, \cdot)$. By a commutative semiring $(S, +, \cdot)$ we mean one for which both semigroups $(S, +)$ and $(S, \cdot)$ are commutative.

**Definition 2.2.** Let $(S, +, \cdot)$ be a semiring and $\leq$ a partial order on $S$. Then $(S, +, \cdot, \leq)$ is called a partially ordered (p. o.) semiring if the following two conditions are satisfied:

1. $(S, +, \leq)$ is a p. o. semigroup, i. e. $M^+(\leq) = S$ holds and
2. $P^+(\leq)$ is contained in $M^+(\leq)$.

If merely (2.3) is satisfied, we call $(S, +, \cdot, \leq)$ a weakly p. o. semiring (cf. [9]). If $\leq$ is a total order, we get the concept of a (weakly) t. o. semiring.

If there exists a zero $o$ in a weakly p. o. semiring $(S, +, \cdot, \leq)$, clearly $P^+(\leq) = \{p \in S \mid o \leq p\}$ holds. Finally, we define extensions of partial orders.

**Definition 2.3.** Let $(S, \leq_l)$ and $(T, \leq_T)$ be partially ordered sets satisfying $S \subseteq T$. Then $\leq_T$ is called an extension of $\leq_l$ if

$$a \leq b \Rightarrow a \leq_T b \text{ for all } a, b \in S$$

holds, and $\leq_T$ is called a strict extension of $\leq_l$ if even

$$a \leq b \iff a \leq_T b \text{ for all } a, b \in S$$

is true. If $\leq$ and $\leq'$ are any two partial orders on the set $S$ satisfying

$$a \leq b \Rightarrow a \leq' b \text{ for all } a, b \in S,$$

then $\leq'$ is called an internal extension of $\leq$.

So for every extension $\leq_T$ of $\leq$ the restriction $\leq' = \leq_T \cap (S \times S)$ of $\leq_T$ on $S$ is an internal extension of $\leq$, and $\leq_T$ is a strict extension of $\leq'$. Therefore in Sections 4 and 5 of this paper we will investigate extensions of partial orders in