INTEGRATION OF D-DIMENSIONAL COSMOLOGICAL MODELS WITH TWO FACTOR SPACES BY REDUCTION TO THE GENERALIZED EMDEN–FOWLER EQUATION

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The D-dimensional cosmological model on the manifold $M = R \times M_1 \times \cdots \times M_n$, describing the evolution of Einstein factor spaces $M_i$ in the presence of a multicomponent perfect fluid source, is considered. The barotropic equation of state for the mass-energy densities and pressures of the components is assumed in each space. Where the number of non-Ricci-flat factor spaces and the number of perfect fluid components are both equal to two, the Einstein equations for the model are reduced to the generalized Emden–Fowler (second-order ordinary differential) equation, which has been recently investigated by Zaitsev and Polyanin using discrete-group analysis. We generate new integrable cosmological models using the integrable classes of this equation and present the corresponding metrics. The method is demonstrated for the special model with Ricci-flat spaces $M_1$ and $M_2$ and a two-component perfect fluid source.

1. Introduction

We develop a multidimensional generalization [1–14] of the standard Friedman–Robertson–Walker cosmological model for studying the early Universe. If additional dimensions of the space–time manifold exist in reality, they could be dynamically important only at an early evolutionary stage of the Universe. In multidimensional cosmology (see, e.g., [1–30]), it is usually assumed that quantum processes at the beginning of this stage result in a topological separation of the multidimensional space–time into an external three-dimensional space and an additional internal space (or spaces). Then, the space–time acquires the topology $M = R \times M_1 \times \cdots \times M_n$, where $R$ is the time axis, one part of the manifolds $M_1, \ldots, M_n$ is a three-dimensional external space, and the other part constitutes the internal spaces. The internal spaces are usually compact; however, models with noncompact internal spaces have also been discussed [11, 31–33].

The subsequent evolution of the multidimensional Universe admits a classical description in terms of the multidimensional Einstein equations. The integration of these equations is the main result of the present paper. The present world is four-dimensional, which means that the internal spaces are extremely small and inaccessible for experiment. This contraction of the internal spaces together with the expansion of the external space is described by some multidimensional cosmological models (the first such model was proposed in [18]) and is called dynamic compactification.

We consider a mixture of several perfect fluid components as the source of the gravitational field in the multidimensional Einstein equations. Such multicomponent systems are often used in four-dimensional cosmology and are suitable for describing the early stages of the Universe [34].

The paper is organized as follows. In Sec. 2, we describe the general multidimensional cosmological model and obtain the Einstein equations in the form of the Lagrange–Euler equations resulting from a Lagrangian. We develop the n-dimensional vector formalism for integrating the equations of motion and present a list of all known integrable models. In Sec. 3, we propose a method for obtaining a new class of integrable models on the manifold $M = R \times M_1 \times M_2$. This method is based on reducing the Einstein equations to the generalized Emden–Fowler (second-order ordinary differential) equation. The method is useful for any two-component model on the manifold $M = R \times M_1 \times M_2$. The total number of components is equal to the number of non-Ricci-flat spaces plus the number of perfect fluid components. Using the...
integrable classes of the generalized Emden–Fowler equation recently obtained by Zaitsev and Polyanin, we find new integrable cosmological models and their metrics. This method is applied to models with the Ricci-flat spaces $M_1$ and $M_2$ and a two-component perfect fluid in Sec. 4.

2. The model and the equations of motion

In the framework of a cosmological model with $n$ factor spaces, the $D$-dimensional space–time manifold $M$ is the product of the time axis $R$ and $n$ manifolds $M_1, \ldots, M_n$, i.e.,

$$M = R \times M_1 \times \cdots \times M_n. \quad (2.1)$$

The product of one part of these manifolds gives the external three-dimensional space while the remaining part pertains to the internal spaces. We assume these internal spaces to be compact. Further, for generality, we assume that the dimensions $N_i = \dim M_i$ for $i = 1, \ldots, n$ are arbitrary.

The manifold $M$ has the metric

$$g = -e^{2\gamma(t)} dt \otimes dt + \sum_{i=1}^{n} \exp[2x^i(t)]g^{(i)}, \quad (2.2)$$

where $\gamma(t)$ is an arbitrary function determining the time scale $t$ and $g^{(i)}$ is the metric on the manifold $M_i$. Let the manifolds $M_1, \ldots, M_n$ be Einstein spaces, i.e.,

$$R_{k_l, [g^{(i)}]} = \lambda_i g_{k_l, [g^{(i)}]}, \quad k_l, l_i = 1, \ldots, N_i, \quad i = 1, \ldots, n, \quad (2.3)$$

where $\lambda_i$ is constant. In the case where $M_i$ is a space with the constant Riemann curvature $K_i$, the constant $\lambda_i$ is $\lambda_i = K_i (N_i - 1)$ (where $N_i > 1$).

From (2.3), we find nonzero components of the Ricci tensor for metric (2.2) (see [10])

$$R^0_0 = e^{-2\gamma} \left( \sum_{i=1}^{n} N_i (\dot{x}^i)^2 + \ddot{\gamma}_0 - \dot{\gamma}_0 \dot{\gamma}_0 \right), \quad (2.4)$$

and

$$R^{m_i}_{n_i} = \left\{ \lambda_i \exp[-2x^i] + [\ddot{x}^i + \dot{x}^i(\dot{\gamma}_0 - \dot{\gamma})]e^{-2\gamma} \right\} \delta^m_{n_i}, \quad (2.5)$$

where

$$\gamma_0 = \sum_{i=1}^{n} N_i \dot{x}^i. \quad (2.6)$$

The indices $m_i$ and $n_i$ in (2.4) and (2.5) for $i = 1, \ldots, n$ run from $D - \sum_{j=1}^{n} N_j$ to $D - \sum_{j=1}^{n} N_j + N_i$, where $D = 1 + \sum_{i=1}^{n} N_i = \dim M$.

We consider a multicomponent perfect fluid to be the source of the gravitational field. The energy momentum tensor of such a source in the observer reference frame is

$$T^M_N = \sum_{\mu=1}^{m} T^M_N^{(\mu)} \quad (2.7)$$

and

$$T^M_N^{(\mu)} = \text{diag}(-\rho^{(\mu)}(t), p^{(\mu)}_1(t)\delta_{11}^{k_1}, \ldots, p^{(\mu)}_m(t)\delta_{mm}^{k_m}). \quad (2.8)$$

We assume that the barotrophic equation of state

$$p^{(\mu)}_i(t) = (1 - h_i^{(\mu)})\rho^{(\mu)}(t), \quad \mu = 1, \ldots, m. \quad (2.9)$$