FREE ENERGY OF THE TWO-DIMENSIONAL $U(n)$-GAUGE FIELD THEORY ON THE SPHERE

A. B. de Monvel$^1$ and M. V. Shcherbina$^2$

The partition function of the two-dimensional $U(n)$-gauge field theory in the limit $n \to \infty$ is rigorously derived. Recent studies in the theory of random matrices combined with the traditional tools of statistical mechanics were the stimuli for the methods used and the results obtained.

1. Introduction

The two-dimensional non-Abelian gauge field theory (QCD$_2$) is a useful model in which to study the mathematical structure of physically relevant theories in spaces with a realistic number of dimensions (string aspects, confining phase, behavior of various invariant quantities, etc.). An important object for non-Abelian theories is the partition function (more generally, the Wilson loop expectations) on two-dimensional surfaces. This function is the kernel (or its trace) of the heat equation on a gauge group manifold as is to be expected from the path integral approach. In many interesting cases, such functions can be explicitly written in terms of irreducible representations of a gauge group. This has resulted in numerous studies of the QCD$_2$ and related topics.

In [1], the partition function on the sphere corresponding to the $U(n)$-gauge theory was analyzed in the large $n$ (planar) limit. It was shown that in this limit, a phase transition of the third kind occurs when the area $A$ of the sphere is varied. Because the coupling constant $g$ enters the partition function only in the combination $g^2 A$, the transition can be interpreted as a certain change in the large-$n$ asymptotic behavior in the strong-coupling regime in contrast to the (trivial) weak-coupling regime. (See [1-3] for additional references and for discussion of the QCD aspects of this transition.)

In the present paper, we rigorously derive the free energy found in [1]. Our derivation is based on traditional ideas in statistical mechanics. Namely, we use the equivalence of the canonical and grand canonical ensembles and the mean-field limit.

The partition function studied in [1] has the form of a sum over all irreducible representations $R$ of the group $U(n)$:

$$Z(n, A) = \sum_R (\dim R)^2 e^{-A \frac{n}{2} C_2(R)},$$

where $\dim R$ is the dimension of the representation $R$, $C_2(R)$ is the eigenvalue of the quadratic Casimir operator (the Laplacian), and $A$ is the area of the sphere. Using the standard parameterization of irreducible representations of $U(n)$ by their weights, we can write partition function (1.1) in the form [1]

$$Z(n, A) = \frac{1}{n!} e^{-\frac{n}{2}(n^2 - 1)} \sum_{h_1, \ldots, h_n = -\infty}^{\infty} \exp \left\{ -\frac{A n}{2} \sum_{i=1}^{n} \left( \frac{h_i}{n} \right)^2 \right\} \prod_{1 \leq i < j \leq n} (h_i - h_j)^2,$$

where the summation is over all integers $h_1, \ldots, h_n$. It is easy to see that up to the trivial factor

$$\frac{1}{n!} e^{-\frac{n}{2}(n^2 - 1)},$$

1Laboratoire de Physique Mathématique et Géométrie, Université Paris VII, Paris, France.
2Institute for Low Temperature Physics, Kharkov, Ukraine.

this expression is similar to that for the partition function of the unitary invariant set of Hermitian random matrices [4]

\[ Z_n = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp \left\{ -n \sum_{i=1}^{n} V(x_i) \right\} \prod_{1 \leq i < j \leq n} (x_i - x_j)^2 \prod_{i=1}^{n} dx_i \]  

(1.3)

with \( V(x) = Ax^2/2 \). Indeed, if we replace summation in (1.2) by integration, we obtain expression (1.3).

The asymptotic behavior of \( Z_n \) was found by physicists many years ago (see [5, 6]) and with mathematical rigor recently in [7]. However, the direct application of these methods leads to the correct expression (1.2) only for small \( A \) (the weak-coupling phase). We thus encounter a new phase transition that does not occur in the continuous version of (1.2), i.e., in the matrix model. To understand the mathematical nature of this phase transition, we calculate the “toy” partition functions

\[ \hat{Z}_n^a = \int_{x_i \neq x_j} \exp \left\{ -n \sum_{i=1}^{n} V(x_i) \right\} \prod_{i=1}^{n} dx_i, \]  

(1.4)

where we remove the “interaction” terms

\[ \prod_{i<j} (h_i - h_j)^2 \quad \text{and} \quad \prod_{i<j} (x_i - x_j)^2, \]

and use the simpler restrictions

\[ x_i \neq x_j \quad \text{or} \quad h_i \neq h_j, \quad i \neq j. \]  

(1.5)

The continuous partition function \( \hat{Z}_n^a \) from (1.4) is factorized

\[ \hat{Z}_n^a = \prod_{i=1}^{n} \int e^{-nV(x_i)} dx_i. \]

The discrete partition function \( Z_n^a \) generally cannot be written in this way, i.e., even asymptotically for large \( n \), the equality

\[ Z_n^a = \prod_{i=1}^{n} \sum_{h_i} e^{-nV\left(\frac{h_i}{n}\right)} \]

is valid only for a special choice of \( V(x) \) corresponding to the “weak-coupling phase.” This phenomenon occurs because condition (1.5) does not influence the integral in the continuous case whereas in the discrete case, this condition is important, especially in the phase transition. This problem is comparable to the problem of computing the free energy for the ideal Bose gas where the difference between the discrete and continuous case (summation and integration) results in Bose–Einstein condensation.

Therefore, to find the asymptotic behavior of \( Z_n^a \), we need to use a technique that allows condition (1.5) to be automatically taken into account. Such a technique is well known in statistical mechanics. It is the usual method for proving the equivalence of the canonical and grand canonical ensembles: introducing the chemical potential. We prove the following theorem, which is a rigorous version of the main result in [1], analogously.