EXACT SOLUTIONS OF THE NONSTATIONARY SCHRÖDINGER EQUATION

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On the basis of exactly solvable stationary models for the Schrödinger equation, we develop a procedure for solving the nonstationary Schrödinger equation in an explicit analytic form. We investigate the formation of the nonadiabatic geometric phase during cyclic evolution of a quantum system.

1. Introduction

We assume that the time evolution of the state $|\Psi(t)\rangle$ of a quantum system is governed by the Schrödinger equation

$$i\hbar \frac{d|\Psi(t)\rangle}{dt} = H(t)|\Psi(t)\rangle. \quad (1)$$

Using the evolution operator $U(t) = U(t,0)$, we can write the solution of Eq. (1) as

$$|\Psi(t)\rangle = U(t)|\Psi(0)\rangle. \quad (2)$$

In this case, the evolution operator satisfies the equation

$$i\hbar \frac{dU(t)}{dt} = H(t)U(t),$$

whose solution is conventionally written in the symbolic form

$$U(t) = T \exp \left( -\frac{i}{\hbar} \int_0^t H(t') dt' \right),$$

where $T$ is the chronological operator. The solution to the nonstationary equation (1) can also be obtained with the general unitary transformation $\hat{S}(t) = \hat{S}^{-1}(t)$,

$$|\Psi(t)\rangle = \hat{S}(t)|\tilde{\psi}(t)\rangle. \quad (3)$$

Because this transformation is unitary and conserves the form of Eq. (1), the induced transformation formulas for the initial Hamiltonian and the evolution operator are given by

$$H(t) \rightarrow \hat{H}(t) = \hat{S}^\dagger(t)H(t)\hat{S}(t) - i\hbar \frac{d}{dt}\hat{S}(t), \quad (4)$$

$$U(t) \rightarrow \hat{U}(t) = \hat{S}^\dagger(t)U(t)\hat{S}(0). \quad (5)$$

It is easy to see that the action of the unitary transformation $\hat{S}(t)$ is very similar to that of the non-Abelian gauge transformations in particle physics. The question is how are these quantum gauge

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transformations useful in solving Eq. (1). We can conveniently introduce a gauge transformation \( S(t) = S(t) \) such that the induced equation resulting from Eq. (1) takes the form

\[
\frac{i\hbar}{\partial t} \frac{d|\Phi(t)\rangle}{dt} = \bar{H}|\Phi(t)\rangle,
\]

(6)

where the induced Hamiltonian \( \bar{H}(t) \) given by relation (4) is independent of time, i.e., \( d\bar{H}/dt = 0 \) (see, e.g., [1, 2]). The operator \( S(t) \) then satisfies the equation

\[
\dot{S}^\dagger H S + \dot{S}^\dagger \frac{\partial H}{\partial t} S + S^\dagger H \dot{S} - i\hbar \dot{S}^\dagger \dot{S} - i\hbar \dot{S}^\dagger \dot{S} = 0,
\]

(7)

where the designation \( \dot{S} = dS/dt \) is used.

For the considered time-independent potential \( \bar{H} \), the solution of Eq. (6),

\[
|\Phi(t)\rangle = \tilde{U}(t,0)|\Phi(0)\rangle \equiv \exp \left( -\frac{i}{\hbar} \bar{H} t \right) |\Phi(0)\rangle,
\]

(8)

may be written as

\[
|\Phi(t)\rangle = \exp \left( -\frac{i}{\hbar} \mathcal{E} t \right) |\varphi(\mathcal{E})\rangle.
\]

(9)

and is properly defined by solutions of the stationary problem

\[
\bar{H}|\varphi(\mathcal{E})\rangle = \mathcal{E}|\varphi(\mathcal{E})\rangle.
\]

(10)

That is, the functions \( |\varphi(\mathcal{E})\rangle = |\Phi(t = 0)\rangle \) are eigenfunctions of the operator \( \bar{H} \) and are also the initial state functions for Eq. (6). Because the evolution operator for \( \Phi(t) \) in (8) takes the form

\[
\tilde{U}(t) = \exp \left( -\frac{i}{\hbar} \bar{H} t \right),
\]

it is easy to see that, as follows from (5), the evolution operator \( U(t) \) and the transformation \( S(t) \) are connected by the relation

\[
U(t) = S(t) \exp \left( -\frac{i}{\hbar} \bar{H} t \right) S^\dagger(0).
\]

(11)

It is easy to verify that with \( t = 0 \) in Eq. (4) for \( S(t) \), the time-independent Hamiltonian \( \bar{H} \) and the Hamiltonian \( H(t) \) at the initial time \( t = 0, H_0 = H(0) \), are connected by the relation

\[
\bar{H} = S^\dagger(0) H_0 S(0) - i\hbar S^\dagger(0) \dot{S}(0).
\]

(12)

Taking relation (5) into account, we obtain the relation between the Hamiltonians at the initial time and an arbitrary time

\[
H(t) = S(t) S^\dagger(0) H_0 S(0) S^\dagger(t).
\]

(13)

Thus, with the use of unitary transformations that generate a time-independent potential, the solution of Eq. (1) is reduced to the solution of Eq. (10) with respect to \( |\varphi(\mathcal{E})\rangle \) and the solution of Eq. (7) with respect to \( S(t) \). It is customary that the solutions of quantum scattering problems are related to the solutions of one-channel or multi-channel systems of equations (10) which characterize stationary processes. In the general case, however, solving Eq. (7) with respect to \( S(t) \) for a given \( H(t) \) is likely more complicated than solving the starting equation (1).

Following [3], we propose that the extensive starting family of time-independent Hamiltonians should comprise Hamiltonians such that the corresponding stationary Schrödinger equation (10) has analytic solutions. In this case, for given operators \( S(t) \) of a definite form, we can construct the corresponding time-dependent Hamiltonians and the exact solutions of the nonstationary Schrödinger equation (1).