RESEARCH ARTICLE

TENSOR PRODUCT OF PARTIALLY-ADDITIVE MONOIDS

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ABSTRACT

Partially-additive monoids (pams) were introduced by Arbib and Manes in order to provide an algebraic semantics for programming languages. In this paper, we prove that the category \( \text{Pam} \) of pams and additive maps is a closed category whose monoids are partially-additive semirings.

We follow the tensor product construction of R. Guitart [7] for categories of algebras which generalize the case of modules. Nevertheless, the problem here is more difficult owing to the fact that pams are partial algebras rather than algebras. Thus, we have to make some modifications to Guitart's approach.

INTRODUCTION

In the 1977 ACM Turing Lecture, John Backus said: computer "programs can be expressed in a language that has an associated algebra. This algebra can be used to transform programs and so solve equations whose 'unknowns' are programs, in much the same way one solves equations in high school algebra" [4,p.619].

Following this, partially-additive monoids (pams)
were introduced by Arbib and Manes [1,2,3]. The partially-additive semantics of recursion supplies algebraic equations to be solved in the sense of Backus. Here, we meet additive and n-additive maps, and partially-additive semirings. We refer the reader to [1,2,3] for further computer science motivations for the concepts of this paper.

In [10] Manes and Benson study the inverse semigroup of some partially-additive semirings. We refer the reader there to see the relationship between pams and semigroups.

This article is an outgrowth of partially-additive semantics. Here we deal with the category $\text{Pam}$ of pams as objects and additive maps as arrows. We prove that this is a closed category whose monoids are the partially-additive semirings in the same way the category of abelian groups is closed and their monoids are ordinary rings. Computer science applications will be dealt with elsewhere.

The core of the paper is the construction of the tensor product. To do this we follow the steps to achieve the tensor product in modules over a commutative ring. This is, essentially, the approach of Fraser [6] to build the tensor product of semilattices and distributive lattices. Let us recall that pams include many of these structures. In any case the problem here is certainly more difficult than that for modules or lattices owing to the fact that pams are partial algebras rather than algebras, that is to say, neither maps nor sums are always defined.

We deal with pams as a category of algebras of an algebraic theory (monad or triple) [5] and we follow the tensor product construction of R. Guitart [7] for such categories of algebras. But $\text{Pam}$ is not such a category, and thus we have to make some modifications to Guitart's approach.

Roughly speaking, pams are commutative monoids for which (finite or infinite) sums are not always defined. Thus, in order to present pams as algebras, we take $\text{Pfn}$ as the base category, where $\text{Pfn}$ is the category whose objects are sets and whose morphisms are partially-defined functions (partial transformations). Hence, the well-