AN INITIAL-VALUE METHOD FOR FREDHOLM INTEGRAL EQUATIONS WITH DISPLACEMENT KERNELS: REFLECTION FUNCTIONS (*)

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SUMMARY - The Fredholm integral equation

\[ u(t) = g(t) + \lambda \int_{0}^{t} k(t, y) u(y) \, dy, \]

where the kernel is \( k(t, y) = k(|t - y|) \), is converted into an initial-value problem. The basic idea is to regard the interval length as variable, while \( t \) is held fixed. Certain auxiliary functions are introduced, one of them being analogous to the reflection function of transport theory. The complete system of differential equations is suitable for numerical solution.

1. Introduction.

In several earlier Memoranda (1, 2, 3, 4) it has been shown that the general Fredholm integral equation of the second kind can be reduced to an initial value problem. This is of analytical interest and of numerical utility in view of the modern computer's ability to integrate large systems of ordinary differential equations subject to initial conditions.

The case of a general kernel is treated in Ref. 4. The case in which the kernel is a function of the absolute value of the difference of the arguments is treated in Ref. 2. This Memorandum provides an alternative approach to this latter problem. Typical numerical results are available in Refs. 5-8.

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The importance of such integral equations in atmospheric physics is stressed, e. g., in Ref. 9.

2. Analytical theory.

Let us consider the Fredholm integral equation,

\[ u(t) = g(t) + \lambda(t) \int_0^c k(t, y) u(y) \, dy, \quad 0 \leq t \leq c. \]

It is assumed that the kernel \( k(t, y) \) depends only on the difference of its arguments, and is an even function,

\[ k(t, y) = k(t - y) = k(y - t), \]

and is representable in the form

\[ k(r) = \int_0^b e^{-rz} dW(z), \quad r \geq 0. \]

This case is of great importance in atmospheric physics\(^{(9)}\) and elsewhere.

In the theory which follows, the interval of integration is variable, while \( t \) is held fixed. For this reason, Eq. (1) is written

\[ u(t, x) = g(t) + \lambda(t) \int_0^x k(t - y) u(y, x) \, dy, \quad 0 \leq t \leq x. \]

The solution \( u(t, x) \) is uniquely defined for \( |\lambda(t)| \) sufficiently small. An exact initial-value problem for \( u(t, x) \) is to be derived. We proceed formally.

Differentiation of Eq. (4) with respect to \( x \) yields

\[ u_x(t, x) = \lambda(t) k(t - x) u(x, x) + \lambda(t) \int_0^x k(t - y) u_x(y, x) \, dy, \]

which is viewed as an integral equation for \( u_x(t, x) \). Introduce the function \( \Phi(t, x) \) as the solution of the integral equation

\[ \Phi(t, x) = \lambda(t) k(x - t) + \lambda(t) \int_0^x k(t - y) \Phi(y, x) \, dy. \]