ABSTRACT - The aim of this paper is to discuss the behaviour of the numerical solution of systems of nonlinear reaction-diffusion equations with homogeneous Neumann boundary conditions. We show that, under certain conditions, the solution obtained using known finite difference methods reproduces the behaviour of the exact solution. In particular we prove that the numerical solution decays as time increases to a spatially homogeneous vector, which is a suitably «weighted» mean value of the numerical solution itself.

1. Introduction.

Problems in populations genetics, nerve pulse transmission, ecology, epidemics and several other biological questions have motivated a growing interest in semilinear parabolic equations [8].

Here we consider reaction-diffusion systems whose general form is

\[
\frac{\partial u_i}{\partial t} = d_i \Delta u_i + f_i(u_1, \ldots, u_m) \quad \text{in } \Omega \times R^+
\]

for \(i = 1, 2, \ldots, m\), where \(\Omega\) is a bounded spatial domain of \(R^n\) and \(d_i\) are positive constants. The solution \(u_i\) is subject to initial conditions:

\[
u_i(x, 0) = u_{0i}(x) \quad x \in \Omega
\]
and to homogeneous Neumann boundary conditions:

\[
\frac{\partial u_i}{\partial n} = 0 \quad \text{on } \partial \Omega \times R^+
\]

where \( \partial \Omega \) is the smooth boundary of \( \Omega \) and \( \frac{\partial}{\partial n} \) denotes the outward normal derivative at a point of \( \partial \Omega \). In this context the function \( u_i(x,t) \) represents the density of the \( i \)-th interacting species and \( d_i \) its diffusion rate; the terms involving the Laplace operator \( \Delta \) represent diffusion processes, while the function \( f_i \) represents the nonlinear interaction terms. The boundary condition (1.1c) indicates that the habitat \( \Omega \) is isolated during the evolution of the system. As a consequence of (1.1c) every solution of the ordinary differential system:

\[
\frac{du_i}{dt} = f_i(u_i, \ldots, u_m) \quad i = 1, \ldots, m
\]

linked with (1.1a) may be viewed as a spatially homogeneous solution of (1.1).

This paper extends results of Conway, Hoff and Smoller on the decay to spatially homogeneous states for solutions of reaction-diffusion systems to an approximating difference scheme. In [5] a parameter \( S_c \) was introduced such that when it is positive (large diffusion) the solution of (1.1) decays exponentially to a spatially homogeneous function of time, which is exactly its average over \( \Omega \). Such a parameter is given by

\[
S_c = d \lambda - M
\]

where \( \lambda \) is the smallest nonzero eigenvalue of \( -\Delta \) on \( \Omega \) with homogeneous Neumann boundary conditions, and

\[
d = \min_i d_i, \quad M = \max_i |dF(u)|, \quad \text{with } F(u) = (f_1(u), \ldots, f_m(u))', \quad \text{for } u \text{ in an invariant region for (1.1).}
\]

We show that it is possible to associate an analogous parameter \( \sigma \) such that when it is positive also the numerical solution of (1.1) decays to a spatially homogeneous vector. This vector is a «weighted» mean value of the numerical solution, the weight being due to the discretization of the Neumann boundary condition (1.1c). Moreover we shall see that if the nonlinear term \( f_i(u) \) verifies a monotonicity condition, the numerical solution decays even if the diffusion is not large. Finally, we prove that the asymptotic behaviour of the mean value is given by the solution of corresponding scheme for (1.2).

The paper is organized as follows: in Section 2 we give a mathematical formulation of our problem and we consider a family of nonlinear finite difference schemes and a family of linearized schemes to solve the system (1.1) numerically; in Section 3 we prove the main results about the asymptotic behavior of the numerical schemes introduced in Section 2; finally in Section 4 we consider an epidemic model to test our results.