THE SCHWARZ ALGORITHM FOR MULTIDOMAIN SPECTRAL APPROXIMATION OF ELLIPTIC PROBLEMS

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ABSTRACT - A multidomain Schwarz method is developed for the spectral approximation of a model elliptic problem. The spectral behavior of the error propagation operator of the algorithm is the same as in the Dirichlet-Neumann method presented in [11]. However, in the Schwarz method only the Dirichlet data have to be interchanged between subdomains, and there is no relaxation parameter to be chosen.

1. Introduction

The Schwarz alternating method was introduced in 1890 (cf. [12]) to solve elliptic problems of certain class. It is based on the concept of the domain decomposition (with overlapping). Due to development of parallel computers, these methods are enjoying now a strong success. They are especially interesting for spectral methods which require treating rectangular (or quadrilateral) domains. Another well-known family of the domain decomposition methods is constituted by the so-called substructuring methods, cf. [13, 6, 7]. These algorithms are based on a non-overlapping subdivision \( \Omega_i \) of the region \( \Omega \). They were applied to approximate the numerical solution of elliptic problem discretized by spectral collocation methods, see [11]. As in the finite element case, in the presence of internal cross points, the rate of convergence of this method grows as the logarithm of the number of degrees of freedom. In this work, we show that the error propagation operator of the Schwarz method has the same behavior. However,
the implementation of the Schwarz algorithm is more straightforward since subproblems of the same type (Dirichlet type) are solved in different subdomains and neither relaxation parameter, nor preconditioner are needed.

This paper is organized as follows. In section 2 we introduce the elliptic model problem and describe the multiplicative and additive Schwarz methods in an abstract framework. In section 3 we discretize the model problem by a variational spectral method. In section 4 we describe the basic decomposition of the domain and we analyze the associated multiplicative and additive Schwarz algorithm. Finally, in section 5 we give some clarifying comments about computational aspects of the Schwarz algorithms.

Throughout this paper, we will use the following notation. Let $\Delta$ denote an open interval of the real axis or a polygonal domain in $\mathbb{R}^2$. We consider the first order Sobolev space $H^1(\Delta)$. The space $H^1_0(\Delta)$ stands for the closure in $H^1(\Delta)$ of the space of infinite differentiable functions with compact supports in $\Delta$. We denote by $\| \cdot \|_{0,\Delta}$ and $\| \cdot \|_{\infty,\Delta}$ the $L^2$-norm and $L^\infty$-norm, respectively, of a function defined on $\Delta$, and $\| \cdot \|^2_{0,\Delta} = \| \cdot \|_{0,\Delta}^2 + \| \nabla (\cdot) \|_{0,\Delta}^2$ is the norm in $H^1(\Delta)$.

2. Multiplicative and additive Schwarz method

We begin with a brief presentation of the classical formulation of Schwarz's method in the case of the continuous Poisson equation, cf. [12]. Let $\Omega$ be an open two-dimensional domain. For $f \in L^2(\Omega)$ we consider the following model problem

\[
\begin{aligned}
-\Delta u &= f \quad \text{in } \Omega \\
 u &= 0 \quad \text{on } \partial \Omega
\end{aligned}
\]  

(1)

Let $\Omega_1$ and $\Omega_2$ be two overlapping subregions, whose union is the region $\Omega$. Given an initial guess $u^0$, the iterate $u^{n+1}$ is determined from $u^n$ by updating sequentially the approximate solution in two subregions:

\[
\begin{aligned}
-\Delta u^{n+1/2} &= f \quad \text{in } \Omega_1 \\
 u^{n+1/2} &= u^n \quad \text{on } \partial \Omega_1
\end{aligned}
\]  

(2)

and

\[
\begin{aligned}
-\Delta u^{n+1} &= f \quad \text{in } \Omega_2 \\
 u^{n+1} &= u^{n+1/2} \quad \text{on } \partial \Omega_2
\end{aligned}
\]  

(3)