THE HERMITE INTERPOLATION

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ABSTRACT - The authors obtain new results on Hermite interpolation based on Jacobi and generalized Jacobi zeros in $C^1$ space and prove error estimates in uniform and weighted $L^p$ norms. The paper gives also the state of art on the topic.

1. Introduction

Let $Y = \{y_{n,k} = y_k, k = 1, ..., n, n \in \mathbb{N}\} \subseteq [-1, 1]$ be an infinite matrix of nodes and for every continuous function $f$ on $[-1, 1]$ ($f \in C^0 = C^0([-1, 1])$) denote by $L_n(f)$ the corresponding Lagrange polynomial of degree at most $n - 1$ interpolating the function $f$ at the points $y_k, k = 1, ..., n$.

The operator $L_n$ maps $C^0$ into itself and denoting by $\|f\| = \|f\|_{\infty} = \max_{|x| \leq 1} |f(x)|$ the usual uniform norm, the norm of the operator $L_n(f)$ is defined by

$$\|L_n\|_{\infty} = \sup_{\|f\| = 1} \|L_n(f)\|.$$  

Usually $\|L_n\|_{\infty}$ is called the $n$-th Lebesgue constant of Lagrange interpolation.

Now let $\{p_n^{\alpha,\beta}\}$ be the sequence of orthonormal polynomials corresponding to the Jacobi weight

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and denote by \( x_{k,n} = x_k, k = 1, ..., n \), the zeros of \( p_n^\alpha \beta \) in natural order.

It is well-known that, if the Jacobi parameters \( \alpha, \beta \) vary in a suitable range (namely \( \alpha, \beta \leq - \frac{1}{2} \)), then the order of Lebesgue constants of the Lagrange interpolation process based at the zeros \( x_k, k = 1, ..., n \), of \( p_n^\alpha \beta \) is optimal, i.e.

\[
\|L_n\|_\infty \leq C \log n,
\]

with \( C \) a constant independent of \( n \).

Now consider the case of Hermite interpolation.

Let \( f \) be a differentiable function on \([-1, 1]\) (\( f \in C^1 = C^1([-1, 1]) \)) and denote by \( H_{2n}(f) \) the corresponding Hermite interpolation polynomial of degree at most \( 2n - 1 \) interpolating the given function \( f \) and its derivative \( f' \) at the zeros \( x_k, k = 1, ..., n \), of \( p_n^\alpha \beta \), i.e.

\[
H_{2n}(f; x) = \sum_{k=1}^{n} l_k^0(x) \, v_k(x) \, f(x_k) + \sum_{k=1}^{n} l_k^1(x) \, (x - x_k) \, f'(x_k),
\]

with \( l_k(x) \) the \( k \)-th fundamental Lagrange polynomial,

\[
v_k(x) = 1 + \frac{\lambda_n'(v^\alpha \beta, x)}{\lambda_n(v^\alpha \beta)} (x - x_k),
\]

\( \lambda_n(v^\alpha \beta, x) \) the \( n \)-th Christoffel function, \( \lambda_{n,k}(v^\alpha \beta, x_k) = \lambda_n(v^\alpha \beta, x_k), k = 1, ..., n \), and \( \lambda_{n,k}'(v^\alpha \beta, x_k) = \lambda_n'(v^\alpha \beta, x_k), k = 1, ..., n \).

Obviously \( H_{2n} \) is an operator from \( C^1 \) to \( C^1 \), i.e. \( H_{2n} : C^1([-1, 1]) \to C^1([-1, 1]) \) and if in \( C^1 \) we define as usually the norm

\[
\|f\|_{\infty,1} = \max(\|f\|, \|f'\|),
\]

then the norm of the operator \( H_{2n} \) is

\[
\|H_{2n}\|_{\infty,1} = \sup_{\|f\|_{\infty,1} = 1} \|H_{2n}(f)\|_{\infty,1}.
\]

On the other hand it is easy to get the following classical estimate

\[
\|f - H_{2n}(f)\|_{\infty,1} \leq \text{const} (1 + \|H_{2n}\|_{\infty,1}) E_{2n-2}(f'),
\]