MULTIDIMENSIONAL STRUCTURED MATRICES AND POLYNOMIAL SYSTEMS

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ABSTRACT - We apply and extend some well-known and some recent techniques from algebraic residue theory in order to relate to each other two major subjects of algebraic and numerical computing, that is, computations with structured matrices and solving a system of polynomial equations. In the first part of our paper, we extend the Toeplitz and Hankel structures of matrices and some of their known properties to some new classes of structured (quasi-Hankel and quasi-Toeplitz) matrices, naturally associated to systems of multivariate polynomial equations. In the second part of the paper, we prove some relations between these structured matrices, which extend the classical relations of the univariate case.

1. Introduction

We apply and extend some well-known and some recent techniques from algebraic residue theory in order to relate to each other two major subjects of algebraic and numerical computing, that is, the computations with structured matrices and solving a system of polynomial equations. We also reveal some hidden correlations between these two subjects via the study of the associated operators of multivariate displacement. The latter operators naturally extend the univariate displacement operators, which define Toeplitz and/or Hankel structure of matrices (cf. [1]). In our multivariate case, we generalize such a matrix structure and arrive at the new classes of operators and structured matrices, which include operators and matrices associated to the polynomial systems of equations and which we call quasi-Hankel and quasi-Toeplitz operators and matrices since some well-known properties of Toeplitz and Hankel operators and matrices can be extended to them (see section 2). Due to high importance of computations

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with structured matrices (see e.g. [1]), our study of these matrix classes may be of independent technical interest. In section 3, we recall some basic definitions and facts about algebraic residues, in order to extend classical relations between structured matrices to the multivariate case (section 4).

Next, we will state some definitions. \( R = \mathbb{C}[x_1, \ldots, x_n] \) will denote the polynomial ring in variables \( x_1, \ldots, x_n \) over the complex field \( \mathbb{C} \), and \( L = \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \) will denote the ring of Laurent’s polynomials in the same variables. We will write \( x = (x_1, \ldots, x_n) \) and \( x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \). For a vector \( \alpha = (\alpha_1, \ldots, \alpha_n) \), we will write \( |\alpha| \) to denote the 1-norm of this vector, \( |\alpha| = \sum_{i=1}^{n} |\alpha_i| \). The total degree of a monomial \( c \cdot x^\alpha \), with a coefficient \( c \), is \( |\alpha| \). The total degree of a polynomial \( \sum_\alpha c_\alpha x^\alpha \), with coefficients \( c_\alpha \neq 0 \), is the highest total degree of its monomials. We will write \( |S| \) to denote the cardinality of a set \( S \). \( \text{ops} \) will stand for "arithmetic operations". \( e_i \) will denote the \( i \)-th unit coordinate vector in \( \mathbb{C}^n \).

Our study can be immediately extended from the complex field \( \mathbb{C} \) to the case of any number field of constants having characteristic 0. Furthermore, with the exception of the results based on the interpolation techniques of [3] (cf. proposition 2.4), our study can be extended to the case of any field of constants.

2. Structured Matrices

In this section, we propose a generalization of the structure of Toeplitz, Hankel and Vandermonde matrices to the case of matrices associated with multivariate polynomials having rows and columns indexed by monomials.

**QuasiHankel and quasiToeplitz matrices operators and the associated generating polynomials definitions and a correlation**

**Definition 2.1.** Let \( E \) and \( F \) be two subsets of \( \mathbb{Z}^n \) and let \( M = (m_{\alpha,\beta})_{\alpha \in E, \beta \in F} \) be a matrix whose rows and columns are indexed by the elements of \( E \) and \( F \) respectively.

- \( M \) is an \((E, F)\) quasiHankel matrix iff, for all \( \alpha \in E, \beta \in F \), the entries \( m_{\alpha,\beta} = h_{\alpha+\beta} \) depend only on \( \alpha + \beta \), that is, if for every \( i = 1, \ldots, n \), we have \( m_{\alpha-e_i,\beta+e_i} = m_{\alpha,\beta} \), provided that \( \alpha, \alpha - e_i \in E; \beta, \beta + e_i \in F \); such a matrix \( M \) is associated with the Laurent polynomial \( H_M(x) = \sum_{u \in E-F} h_u x^{-u} \).

- \( M \) is an \((E, F)\) quasiToeplitz matrix iff, for all \( \alpha \in E, \beta \in F \), the entries \( m_{\alpha,\beta} = t_{\alpha-\beta} \) depend only on \( \alpha - \beta \), that is, if for every \( i = 1, \ldots, n \), we have \( m_{\alpha+e_i,\beta+e_i} = m_{\alpha,\beta} \), provided that \( \alpha, \alpha + e_i \in E; \beta, \beta + e_i \in F \); such a matrix \( M \) is associated with the polynomial \( T_M(x) = \sum_{u \in E+F} t_u x^u \).

For \( E = [0, \ldots, m - 1] \) and \( F = [0, \ldots, n - 1] \), definition 2.1 turns into the usual definition of Hankel (resp. Toeplitz) matrices [1].