1. Introduction.

Test matrices have been in use for some time to scrutinize computer algorithms for solving linear algebraic systems and eigenvalue problems; see, e.g., Gregory and Karney [1969]. For the problem of finding roots of algebraic equations, the construction of appropriate test equations has been given less attention. Here we wish to propose two families of algebraic test equations, the first consisting of equations with predominantly complex roots, the second of equations with only real roots.

To be useful for testing purposes, a test equation (of some fixed degree) should have the following characteristics:

(1) All roots of the equation can be calculated directly (i.e., without recourse to a rootfinding algorithm).

(2) The equation contains a parameter (or parameters) which can be used to control the numerical condition of the roots. By varying the parameter(s), the condition number of the worst-conditioned root can be made to range from relatively small to arbitrarily large values.

(3) All coefficients of the equation are integer-valued.

It may not be easy, in practice, to achieve all these characteristics, particularly the last one, if we are interested in relatively large degrees. Even when this is possible, the integer coefficients may become so large as to make exact representation in floating-point arithmetic impossible. Although equations which do not satisfy (3) are less desirable, they are still useful for testing purposes,
provided one takes properly into account the influence of rounding errors in the coefficients upon the results.

Isolated examples of test equations, particularly ill-conditioned ones, have been known for a long time. Perhaps the best-known example, due to Wilkinson [1963], is the equation with roots at 1, 2, ..., n. Some of these roots are relatively well-conditioned, while others are quite ill-conditioned, more so the larger the degree n. (The numerical condition of Wilkinson's equation is analyzed in Gautschi [1973]; see also Gautschi [1978, §4]). The roots of unity lead to another interesting example if one removes half of them and retains only those on the half-circle (Jenkins and Traub [1975]). Our first family of test equations, indeed, is a simple extension of this latter example.

2. A first family of test equations and their numerical condition.

Given an algebraic equation of degree n,

(2.1) \[ p(z) = 0, \quad p(z) = z^n + a_1 z^{n-1} + \ldots + a_{n-1} z + a_n, \]

with complex coefficients \(a_i\), and a simple root \(\zeta\) of (2.1), we adopt as condition number for \(\zeta\) the quantity (Wilkinson [1963, p. 38 ff], Gautschi [1973])

(2.2) \[ \text{cond } \zeta = \frac{\sum_{i=1}^{n} |a_i|}{|\zeta| |p'(\zeta)|}. \]

It measures the sensitivity of \(\zeta\), in terms of relative errors, to small (relative) perturbations in all nonzero coefficients \(a_i\).

Our first family of test equations (with parameter \(\alpha\)) is

(2.3) \[ p_\alpha(z) = 0, \quad p_\alpha(z) = \prod_{k=1}^{n} [z - \zeta_k(\alpha)] = z^n - \sigma_1 z^{n-1} + \ldots + (-1)^n \sigma_n, \]

\[ \zeta_k(\alpha) = e^{(k-1)i\alpha}, \quad 0 < \alpha \leq \frac{2\pi}{n}. \]

If \(\alpha = 2\pi/n\), the \(\zeta_k\) are the \(n\)-th roots of unity, thus \(p_\alpha(z) = z^n - 1\), and we are in the case of a well-conditioned equation, all roots having condition \(1/n\). For \(\alpha = \pi/(n-1)\), we get the roots of unity on a half-circle, which are all relatively ill-conditioned. As \(\alpha \downarrow 0\), the condition deteriorates unboundedly.