THE CONVERGENCE ORDER FOR ITERATIVE MULTIPOINT PROCEDURES

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SUMMARY - In this paper we study the convergence order for iterative multipoint procedures with memory and we determine also upper bounds for such an order. As a particular case, from the formulae we have deduced, we also obtain the convergence orders of iterative multipoint procedures without memory. In addition we fix the conditions which are necessary and sufficient to make such procedures optimal.

Introduction.

The problem of finding an approximation of the root \( \alpha \) in the equation

\[ f(x) = 0 \]

generally changes into that of looking for a fixed point for an iterative procedure. In this paper we are concerned with iterative multipoint procedures.

In section 1 we give first of all some essential definitions, among which there are definitions which classify the multipoint procedures as multipoints without memory, with external memory and with internal memory.

In section 2 we discuss the iterative multipoint procedures with external memory by giving a theorem which allows us to calculate the convergence order of such iterative procedures, and which shows that is possible

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to give a general formula for the calculation of the order, even in the case of iterative multipoint procedures without memory. Also we give a majorization for the convergence order and a theorem of necessary and sufficient conditions for maximizing such an order.

In section 3 we study iterative multipoint procedures with internal memory and we give for these too a practical method for calculating the convergence order and a majorization of this; in addition we give the conditions which are necessary and sufficient for maximizing the convergence order.

Finally the appendix shows some numerical results.

1. Definitions.

In line with Traub [1] we classify the iterative procedures from the information which they demand.

An iterative procedure of the type

\[ x_{n+l} = G(x_n, x_{n+1}, \ldots, x_{n+l-1}) \]

which demands information in the points

\[ x_n, x_{n+1}, \ldots, x_{n+l-1} \]

when \( l > 1 \), is called an iterative one-point procedure with memory. If \( l = 1 \) it is called an iterative one-point procedure without memory.

In the following, we shall assume \( G \in C^q(I_a) \) with \( q \) sufficiently high and \( I_a \) is an open neighbourhood of the fixed point \( a \).

**Definition 1.1.** Given \( k \) iterative (one-point) procedures with or without memory of the type

\[ x_{n+l+1} = G_1(x_{n+1}, x_{n+2}, \ldots, x_{n+l}) \]

\[ x_{n+l+2} = G_2(x_{n+1}, x_{n+2}, \ldots, x_{n+l+1}) \]

\[ \vdots \]

\[ x_{n+l+k} = G_k(x_{n+1}, x_{n+2}, \ldots, x_{n+l+k-1}) \]